Sheaf Theory

A categorical approach

Alessandro Pizzotti

Contents

Appendix A

Some facts on categories and limits

A.1 Comma categories

Definition A.1.1. Let $\mathbf{C}$ be a category and $\mathbf{c} \in \mathcal{O}(\mathbf{C})$.
The ***comma category*** $(\mathbf{c}\downarrow\mathbf{C})$ ***of objects of*** $\mathbf{C}$ ***under*** $\mathbf{c}$ is defined by

$$\mathcal{O}((\mathbf{c}\downarrow\mathbf{C})) = \{(u,\mathbf{d})|\mathbf{d} \in \mathcal{O}(\mathbf{C}), u \in \overline{\mathbf{C}}(\mathbf{c},\mathbf{d})\} \tag{A.1}$$

and

$$\overline{(\mathbf{c}\downarrow\mathbf{C})}\,((u_1,\mathbf{c}_1),(u_2,\mathbf{c}_2)) = \{f \in \overline{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)|f \circ u_1 = u_2\}; \tag{A.2}$$

the ***comma category*** $(\mathbf{C}\downarrow\mathbf{c})$ ***of objects of*** $\mathbf{C}$ ***over*** $\mathbf{c}$ is defined by

$$\mathcal{O}((\mathbf{C}\downarrow\mathbf{c})) = \{(\mathbf{d},u)|\mathbf{d} \in \mathcal{O}(\mathbf{C}), u \in \overline{\mathbf{C}}(\mathbf{d},\mathbf{c})\} \tag{A.3}$$

and

$$\overline{(\mathbf{C}\downarrow\mathbf{c})}\,((\mathbf{c}_1,u_1),(\mathbf{c}_2,u_2)) = \{f \in \overline{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)|u_2 \circ f = u_1\}. \tag{A.4}$$

Definition A.1.2. Let $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ be a functor and $\mathbf{d} \in \mathcal{O}(\mathbf{D})$.
The ***comma category*** $(\mathbf{d}\downarrow\mathrm{F})$ ***of objects of*** $\mathbf{C}$ F***-under*** $\mathbf{d}$ is defined by

$$\mathcal{O}((\mathbf{d}\downarrow\mathrm{F})) = \{(u,\mathbf{c})|\mathbf{c} \in \mathcal{O}(\mathbf{C}), u \in \overline{\mathbf{D}}(\mathbf{d},\mathrm{F}(\mathbf{c}))\} \tag{A.5}$$

and

$$\overline{(\mathbf{d}\downarrow\mathrm{F})}\,((u_1,\mathbf{c}_1),(u_2,\mathbf{c}_2)) = \{f \in \overline{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)|\mathrm{F}(f) \circ u_1 = u_2\}; \tag{A.6}$$

the ***comma category*** $(\mathrm{F}\downarrow\mathbf{d})$ ***of objects of*** $\mathbf{C}$ F***-over*** $\mathbf{d}$ is defined by

$$\mathcal{O}((\mathrm{F}\downarrow\mathbf{d})) = \{(\mathbf{c},u)|\mathbf{c} \in \mathcal{O}(\mathbf{C}), u \in \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}),\mathbf{d})\} \tag{A.7}$$

and

$$\overline{(\mathrm{F}\downarrow\mathbf{d})}\,((\mathbf{c}_1,u_1),(\mathbf{c}_2,u_2)) = \{f \in \overline{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)|u_2 \circ \mathrm{F}(f) = u_1\}. \tag{A.8}$$

A.2 Universal arrows and limits

Proposition A.2.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$*. There is a functor* $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}}$ *such that for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$

$$\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}}(\mathbf{c}) = \mathbf{d} \tag{A.9}$$

and for $f \in \mathcal{M}(\mathbf{C})$

$$\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}}(f) = \mathrm{id}_{\mathbf{d}}\,. \tag{A.10}$$

Proof. Routine check. ✠

Definition A.2.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, $\mathbf{d} \in \mathcal{O}(\mathbf{D})$. The functor $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}}$ is called the ***constant functor from*** $\mathbf{C}$ ***to*** $\mathbf{D}$ ***relative to the object*** $\mathbf{d}$.

Proposition A.2.2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $f \in \mathcal{M}(\mathbf{D})$*. There is a natural transformation* $\gamma_f^{\mathbf{C},\mathbf{D}} : \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathrm{dom}(f)} \to \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathrm{cod}(f)}$ *such that for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$

$$\gamma_f^{\mathbf{C},\mathbf{D}}{}_{\mathbf{c}} = f. \tag{A.11}$$

Proof. Routine check. ✠

Definition A.2.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, $f \in \mathcal{M}(\mathbf{D})$. The natural transformation $\gamma_f^{\mathbf{C},\mathbf{D}}$ is called the ***constant natural transformation from*** $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathrm{dom}(f)}$ ***to*** $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathrm{cod}(f)}$ ***relative to the morphism*** f.

Proposition A.2.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathbf{d}_1 \in \mathcal{O}(\mathbf{D})$ $\mathbf{d}_2 \in \mathcal{O}(\mathbf{D})$*. If* $\tau : \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1} \to \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}$ *is a natural transformation, then there exists* $f \in \overline{\mathbf{D}}(\mathbf{d}_1, \mathbf{d}_2)$ *such that* $\tau = \gamma_f^{\mathbf{C},\mathbf{D}}$.

Proof. For $\mathbf{c}_1 \in \mathcal{O}(\mathbf{C})$, $\mathbf{c}_2 \in \mathcal{O}(\mathbf{C})$ and $g \in \hom_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the diagram

$$\begin{array}{ccc} \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1}(\mathbf{c}_1) & \xrightarrow{\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1}(g)} & \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1}(\mathbf{c}_2) \\ \downarrow{\scriptstyle \tau_{\mathbf{c}_1}} & & \downarrow{\scriptstyle \tau_{\mathbf{c}_2}} \\ \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}(\mathbf{c}_1) & \xrightarrow{\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}(g)} & \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}(\mathbf{c}_2) \end{array} \tag{A.12}$$

must commute. But $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1}(\mathbf{c}_1) = \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1}(\mathbf{c}_2) = \mathbf{d}_1$, $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}(\mathbf{c}_1) = \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}(\mathbf{c}_2) = \mathbf{d}_2$, $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_1}(g) = \mathrm{id}_{\mathbf{d}_1}$, $\mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{d}_2}(g) = \mathrm{id}_{\mathbf{d}_2}$, so $\tau_{\mathbf{c}_2} \circ \mathrm{id}_{\mathbf{d}_1} = \mathrm{id}_{\mathbf{d}_2} \circ \tau_{\mathbf{c}_1}$, that is $\tau_{\mathbf{c}_2} = \tau_{\mathbf{c}_1}$. This holds for any $\mathbf{c}_1 \in \mathcal{O}(\mathbf{C})$ and $\mathbf{c}_2 \in \mathcal{O}(\mathbf{C})$, thus $\tau = \gamma_{\tau_{\mathbf{c}_1}}^{\mathbf{C},\mathbf{D}}$. ✠

Proposition A.2.4. *Let* $\mathbf{J}$, $\mathbf{C}$ *be categories. There is a functor* $\Delta_{\mathcal{J}}^{\mathcal{C}} : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ *defined by:*

- $\Delta_{\mathcal{J}}^{\mathcal{C}}(\mathbf{c}) = \mathrm{C}^{\mathbf{J},\mathbf{C}}_{\mathbf{c}}$ *for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$
- $\Delta_{\mathcal{J}}^{\mathcal{C}}(f) = \gamma_f^{\mathbf{J},\mathbf{C}}$ *for* $f \in \mathcal{M}(\mathbf{C})$.

Proof. Routine check. ✠

Definition A.2.3. Let $\mathbf{J}$, $\mathbf{C}$ be categories. The functor Δ^{C}_{J} is called the ***diagonal functor in*** $\mathbf{C}$ ***relative to*** $\mathbf{J}$. The category $\mathbf{J}$ is called the ***index category for the diagonal functor*** Δ^{C}_{J}.

Definition A.2.4. Let $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ be a functor, $\mathbf{d}\in\mathcal{O}(\mathbf{D})$. An ***arrow from*** $\mathbf{d}$ ***to*** F is a pair $(\mathbf{c},u)$ where $\mathbf{c}\in\mathcal{O}(\mathbf{C})$, $u\in\overline{\mathbf{D}}(\mathbf{d},\mathrm{F}(\mathbf{c}))$.

Remark A.2.1. According to Definition A.2.4 a pair $(\mathbf{c},u)$ is an arrow from $\mathbf{d}\in\mathcal{O}(\mathbf{D})$ to the functor $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ if

1. $\mathbf{c}\in\mathcal{O}(\mathbf{C})$
2. $\mathrm{dom}(u)=\mathbf{d}$
3. $\mathrm{cod}(u)=F(\mathbf{c})$.

Notation A.2.1. Let $\mathbf{C}$ be a category, $\mathrm{F}:\mathbf{C}\to\mathbf{Set}$. We will denote with $\mathbf{C}[\mathrm{F}]$ the category defined by

$$\mathcal{O}(\mathbf{C}[\mathrm{F}])=\{(\mathbf{c},x)|\mathbf{c}\in\mathcal{O}(\mathbf{C}),x\in\mathrm{F}(\mathbf{c})\} \tag{A.13}$$

and

$$\overline{\mathbf{C}[\mathrm{F}]}\left((\mathbf{c}_1,x_1),(\mathbf{c}_2,x_2)\right)=\{f\in\overline{\mathbf{C}}\left(\mathbf{c}_1,\mathbf{c}_2\right)|\mathrm{F}(f)(x_1)=x_2\}. \tag{A.14}$$

Definition A.2.5. Let $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ be a functor, $\mathbf{d}\in\mathcal{O}(\mathbf{D})$. Two arrows $(\mathbf{c}_1,u_1)$, $(\mathbf{c}_2,u_2)$ from $\mathbf{d}$ to F are ***isomorphic*** if there is an isomorphism $i:c_1\to c_2$ such that $u_2=F(i)\circ u_1$.

Definition A.2.6. Let $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ be a functor, $\mathbf{d}\in\mathcal{O}(\mathbf{D})$. An arrow $(\mathbf{c},u)$ from $\mathbf{d}$ to F is ***universal*** if and only if for any arrow $(\mathbf{c}_*,u_*)$ from $\mathbf{d}$ to F there is a unique morphism $f:\mathbf{c}\to\mathbf{c}_*$ such that $u_*=\mathrm{F}(f)\circ u$, that is, the diagram

$$\begin{array}{ccc} \mathbf{d} & \xrightarrow{u} & \mathrm{F}(\mathbf{c}) \\ & \searrow^{u_*} & \downarrow{\scriptstyle \mathrm{F}(f)} \\ & & \mathrm{F}(\mathbf{c}_*) \end{array} \tag{A.15}$$

commutes.

Proposition A.2.5. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ *be a functor,* $\mathbf{d}\in\mathcal{O}(\mathbf{D})$. *An arrow* $(\mathbf{c},u)$ *from* $\mathbf{d}$ *to* F *is universal if and only if* $(\mathbf{c},u)$ *is an initial object in the category* $(\mathbf{d}\downarrow\mathbf{F})$.

Proof. Routine check. ✠

Definition A.2.7. Let $\mathrm{F}:\mathbf{C}\to\mathbf{Set}$ be a functor. A ***universal element of*** F is a pair $(\mathbf{c}, x)$, where $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ and $x \in \mathrm{F}(\mathbf{c})$, and for every $\mathbf{d} \in \mathcal{O}(\mathbf{C})$ and $y \in \mathrm{F}(\mathbf{d})$ there is a unique $f : \mathbf{c} \to \mathbf{d}$ such that $y = \mathrm{F}(f)(x)$.

Proposition A.2.6. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ *be a functor,* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$*. An arrow* $(\mathbf{c}, u)$ *from* $\mathbf{d}$ *to* F *is universal if and only if it is a universal element of the functor* $\overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}-)$*.*

Proof. Routine check. ✠

Proposition A.2.7. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{Set}$ *be a functor. A pair* $(\mathbf{c}, x)$ *is a universal element of* F *if and only if for any one-element set* $*$*, if* $u : * \to \mathrm{F}(\mathbf{c})$ *is the map defined by* $u(*) = x$*, then* $(\mathbf{c}, u)$ *is a universal arrow from* $*$ *to* F*.*

Proof. Routine check. ✠

Proposition A.2.8. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{Set}$ *be a functor. A pair* $(\mathbf{c}, x)$ *is a universal element of* F *if and only if it is an initial object in the category* $\mathbf{C}[\mathrm{F}]$*.*

Proof. Routine check. ✠

Proposition A.2.9. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ *be a functor,* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$*. An arrow* $(\mathbf{c}, u)$ *from* $\mathbf{d}$ *to* F *is universal if and only if the maps*

$$\begin{aligned} \phi^{(\mathbf{c},u)}_{\mathbf{c}_*} : \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}_*) &\to \overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(\mathbf{c}_*)) \\ f &\mapsto \mathrm{F}(f) \circ u \end{aligned}$$

define a natural isomorphism $\phi^{(\mathbf{c},u)}$ *between the hom-functors* $\overline{\mathbf{C}}(\mathbf{c}, -)$ *and* $\overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(-))$*.*

Proof. The statement that the map $\phi^{(\mathbf{c},u)}_{\mathbf{c}_*}$ is a bijection for each $\mathbf{c}_*$ is exactly the statement that $(\mathbf{c}, u)$ is a universal arrow from $\mathbf{d}$ to F. Now, if $g \in \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the diagram

$$\begin{array}{ccc} \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}_1) & \xrightarrow{\phi^{(\mathbf{c},u)}_{\mathbf{c}_1}} & \overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(\mathbf{c}_1)) \\ {\scriptstyle \overline{\mathbf{C}}(\mathbf{c},g)}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{D}}(\mathbf{d},\mathrm{F}(g))} \\ \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}_2) & \xrightarrow{\phi^{(\mathbf{c},u)}_{\mathbf{c}_2}} & \overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(\mathbf{c}_2)) \end{array} \tag{A.16}$$

commutes because, for $f \in \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}_1)$

$$\overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(g))\left(\phi^{(\mathbf{c},u)}_{\mathbf{c}_1}(f)\right) = \overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(g))\left(\mathrm{F}(f) \circ u\right) = \mathrm{F}(g) \circ \mathrm{F}(f) \circ u$$

and

$$\phi^{(\mathbf{c},u)}_{\mathbf{c}_2}\left(\overline{\mathbf{C}}(\mathbf{c}, g)(f)\right) = \phi^{(\mathbf{c},u)}_{\mathbf{c}_2}(f \circ g) = \mathrm{F}(f \circ g) \circ u = \mathrm{F}(g) \circ \mathrm{F}(f) \circ u.$$

so $\phi^{(\mathbf{c},u)}$ is a natural transformation.

✠

Definition A.2.8. Let $(\mathbf{c}, u)$ be a universal arrow. The natural isomorphism $\phi^{(\mathbf{c},u)}$ is called the ***natural isomorphism associated to*** $(\mathbf{c}, u)$.

Proposition A.2.10. *Let* $(\mathbf{c}, u)$ *be a universal arrow. Then* $u = \phi_{\mathbf{c}}^{(\mathbf{c},u)}(\mathrm{id}_{\mathbf{c}})$.

Proof. Let $(\mathbf{c}, u)$ be a universal arrow to the functor F. Then $\phi_{\mathbf{c}}^{(\mathbf{c},u)}(\mathrm{id}_{\mathbf{c}}) = \mathrm{F}(\mathrm{id}_{\mathbf{c}}) \circ u = u$ ✠

Notation A.2.2. For a natural transformation $\phi : \overline{\mathbf{C}}(\mathbf{c}, -) \to \mathrm{F}$ set $u^{\phi} = \phi_{\mathbf{c}}(\mathrm{id}_{\mathbf{c}})$.

Proposition A.2.11. *Let* $\mathrm{F}:\mathbf{C}\to\mathrm{D}$ *be a functor,* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$*,* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$*,* ϕ *a natural isomorphism between the functors* $\overline{\mathbf{C}}(\mathbf{c}, -)$ *and* $\overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}-)$*. Then* $(\mathbf{c}, u^{\phi})$ *is a universal arrow from* $\mathbf{d}$ *to* F *and* $\phi = \phi^{(\mathbf{c},u^{\phi})}$.

Proof. Let $f \in \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}_*)$. Since the diagram

$$\begin{array}{ccc} \overline{\mathbf{C}}(\mathbf{c},\mathbf{c}) & \xrightarrow{\phi_{\mathbf{c}}} & \overline{\mathbf{D}}(\mathbf{d},\mathrm{F}(\mathbf{c})) \\ {\scriptstyle \overline{\mathbf{C}}(\mathbf{c},f)}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{D}}(\mathbf{d},\mathrm{F}(f))} \\ \overline{\mathbf{C}}(\mathbf{c},\mathbf{c}_*) & \xrightarrow{\phi_{\mathbf{c}_*}} & \overline{\mathbf{D}}(\mathbf{d},\mathrm{F}(\mathbf{c}_*)) \end{array} \tag{A.17}$$

commutes, in particular $\phi_{\mathbf{c}_*}\left(\overline{\mathbf{C}}(\mathbf{c}, f)(\mathrm{id}_{\mathbf{c}})\right) = \overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}(f))\left(\phi_{\mathbf{c}}(\mathrm{id}_{\mathbf{c}})\right)$, so for each $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ and $f \in \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}_*)$ $\phi_{\mathbf{c}_*}(f) = \mathrm{F}(f) \circ u^{\phi}$. By Proposition A.2.9 $(\mathbf{c}, u^{\phi})$ is a universal arrow from $\mathbf{d}$ to F. ✠

Corollary A.2.1. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ *be a functor,* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$*. An arrow* $(\mathbf{c}, u)$ *from* $\mathbf{d}$ *to* F *is universal if and only if there is a natural bijection* $\phi : \overline{\mathbf{C}}(\mathbf{c}, -) \to \overline{\mathbf{D}}(\mathbf{d}, \mathrm{F}-)$ *and* $u = \phi_{\mathbf{c}}(\mathrm{id}_{\mathbf{c}})$.

Proposition A.2.12. *Any two universal arrows from an object* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$ *to a functor* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ *are isomorphic.*

Proof. Let $(\mathbf{c}_1, u_1)$ and $(\mathbf{c}_2, u_2)$ both be universal arrows from $\mathbf{d}$ to F. Then, since $(\mathbf{c}_1, u_1)$ is universal there is $f : \mathbf{c}_1 \to \mathbf{c}_2$ such that $u_2 = \mathrm{F}(f) \circ u_1$ and, since $(\mathbf{c}_2, u_2)$ is universal, there is $g : \mathbf{c}_2 \to \mathbf{c}_1$ such that $u_1 = \mathrm{F}(g) \circ u_2$. So $u_1 = \mathrm{F}(g) \circ \mathrm{F}(f) \circ u_1 = \mathrm{F}(g \circ f) \circ u_1$ whence $g \circ f = \mathrm{id}_{\mathbf{c}_1}$, because $(\mathbf{c}_1, u_1)$ is universal. Also $u_2 = \mathrm{F}(f) \circ \mathrm{F}(g) \circ u_2 = \mathrm{F}(f \circ g) \circ u_2$ whence $f \circ g = \mathrm{id}_{\mathbf{c}_2}$, because $(\mathbf{c}_2, u_2)$ is universal. So f is an isomorphism. ✠

Lemma A.2.1 (Yoneda)**.** *A natural transformation* $\varphi : \overline{\mathbf{C}}(\mathbf{c}, -) \to \mathrm{F}$ *is completely determined by* u^{φ}*. More specifically, let* $\mathrm{N}^{\mathbf{C}}:\mathbf{Set}^{\mathbf{C}}\times\mathbf{C}\to\mathbf{Set}$ *be the functor defined by*

- $\mathrm{N}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}) = \overline{\mathbf{Set}}(\overline{\mathbf{C}}(\mathbf{c}, -), \mathrm{F})$ *for* $\mathbf{c} \in \mathcal{O}(\mathbf{C}), \mathrm{F} \in \mathcal{O}(\mathbf{Set}^{\mathbf{C}})$
- $\mathrm{N}^{\mathbf{C}}(\tau, f) = \overline{\mathbf{Set}}(\overline{\mathbf{C}}(f, -), \tau)$

and $\mathrm{E}^{\mathbf{C}}:\mathbf{Set}^{\mathbf{C}}\times\mathbf{C}\to\mathbf{Set}$ *the functor defined by*

- $\mathrm{E}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}) = \mathrm{F}(\mathbf{c})$

- $\mathrm{E}^{\mathbf{C}}(\tau, f) = \tau_{\mathrm{cod}\, f}(\mathrm{dom}\, \tau(f))$.

Then there is a natural isomorphism $y : \mathrm{N}^{\mathbf{C}} \to \mathrm{E}^{\mathbf{C}}$ *defined for* $\mathrm{F} \in \mathcal{O}(\mathbf{Set}^{\mathbf{C}})$ *and* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *by*

$$\begin{aligned} y_{\mathrm{F},\mathbf{c}} : \mathrm{N}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}) &\to \mathrm{E}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}) \\ \varphi &\mapsto u^{\varphi} \end{aligned}. \tag{A.18}$$

Proof. That $\varphi : \overline{\mathbf{C}}(\mathbf{c}, -)\mathrm{F} \to$ is completely determined by u^{φ} follows from the commutative diagram

$$\begin{array}{ccc} \overline{\mathbf{C}}(\mathbf{c}, \mathbf{c}) & \xrightarrow{\varphi_{\mathbf{c}}} & \mathrm{F}(\mathbf{c}) \\ \downarrow{\scriptstyle \overline{\mathbf{C}}(\mathbf{c}, f)} & & \downarrow{\scriptstyle \mathrm{F}(f)} \\ \overline{\mathbf{C}}(\mathbf{c}, \mathbf{d}) & \xrightarrow{\varphi_{\mathbf{d}}} & \mathrm{F}(\mathbf{d}) \end{array} \tag{A.19}$$

which yields

$$\varphi_{\mathrm{d}}(f) = \mathrm{F}(f)(u^{\varphi}). \tag{A.20}$$

So $y_{\mathrm{F},\mathbf{c}}$ is a bijection for $\mathrm{F} \in \mathcal{O}(\mathbf{Set}^{\mathbf{C}})$ and $\mathbf{c} \in \mathcal{O}(\mathbf{C})$.
Let's show that y is natural in F. For $\tau : \mathrm{F}_1 \to \mathrm{F}_2$ we have to show that the diagram

$$\begin{array}{ccc} \mathrm{N}^{\mathbf{C}}(\mathrm{F}_1, \mathbf{c}) & \xrightarrow{y_{\mathrm{F}_1,\mathbf{c}}} & \mathrm{E}^{\mathbf{C}}(\mathrm{F}_1, \mathbf{c}) \\ \downarrow{\scriptstyle \mathrm{N}^{\mathbf{C}}(\tau, \mathbf{c})} & & \downarrow{\scriptstyle \mathrm{E}^{\mathbf{C}}(\tau, \mathbf{c})} \\ \mathrm{N}^{\mathbf{C}}(\mathrm{F}_2, \mathbf{c}) & \xrightarrow{y_{\mathrm{F}_2,\mathbf{c}}} & \mathrm{E}^{\mathbf{C}}(\mathrm{F}_2, \mathbf{c}) \end{array} \tag{A.21}$$

commutes. This can be rewritten as

$$\begin{array}{ccc} \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}, -), \mathrm{F}_1) & \xrightarrow{y_{\mathrm{F}_1,\mathbf{c}}} & \mathrm{F}_1(\mathbf{c}) \\ \downarrow{\scriptstyle \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}, -), \tau)} & & \downarrow{\scriptstyle \tau_{\mathbf{c}}} \\ \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}, -), \mathrm{F}_2) & \xrightarrow{y_{\mathrm{F}_2,\mathbf{c}}} & \mathrm{F}_1(\mathbf{c}) \end{array} \tag{A.22}$$

and this commutes because for $\alpha \in \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}, -), \mathrm{F}_1)$

$$\tau_{\mathbf{c}} \circ y_{\mathrm{F}_1,\mathbf{c}}(\alpha) = \tau_{\mathbf{c}}(u^{\alpha}) \tag{A.23}$$

and

$$y_{\mathrm{F}_2,\mathbf{c}} \circ \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}, -), \tau)(\alpha) = y_{\mathrm{F}_2,\mathbf{c}}(\tau \circ \alpha) = (\tau \circ \alpha)_{\mathbf{c}}(\mathrm{id}_{\mathbf{c}}) = \tau_{\mathbf{c}}(u^{\alpha}). \tag{A.24}$$

Let's show that y is natural in $\mathbf{c}$. For $f : \mathbf{c}_1 \to \mathbf{c}_2$ we have to show that the diagram

$$\begin{array}{ccc} \mathrm{N}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}_1) & \xrightarrow{y_{\mathrm{F},\mathbf{c}_1}} & \mathrm{E}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}_1) \\ \downarrow{\scriptstyle \mathrm{N}^{\mathbf{C}}(\mathrm{F}, f)} & & \downarrow{\scriptstyle \mathrm{E}^{\mathbf{C}}(\mathrm{F}, f)} \\ \mathrm{N}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}_2) & \xrightarrow{y_{\mathrm{F},\mathbf{c}_2}} & \mathrm{E}^{\mathbf{C}}(\mathrm{F}, \mathbf{c}_2) \end{array} \tag{A.25}$$

commutes. This can be rewritten as

$$\begin{array}{ccc} \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}_1,-),\mathrm{F}) & \xrightarrow{y_{\mathrm{F},\mathbf{c}_1}} & \mathrm{F}(\mathbf{c}_1) \\ \Big\downarrow{\scriptstyle \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(f,-),\mathrm{F})} & & \Big\downarrow{\scriptstyle \mathrm{F}(f)} \\ \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}_2,-),\mathrm{F}) & \xrightarrow{y_{\mathrm{F},\mathbf{c}_2}} & \mathrm{F}(\mathbf{c}_2) \end{array} \tag{A.26}$$

and this commutes because for $\alpha \in \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(\mathbf{c}_1,-),\mathrm{F})$

$$\mathrm{F}(f) \circ y_{\mathrm{F},\mathbf{c}_1}(\alpha) = \mathrm{F}(f)(u^{\alpha}) \tag{A.27}$$

and

$$\begin{aligned} y_{\mathrm{F},\mathbf{c}_2} \circ \overline{\mathbf{Set}^{\mathbf{C}}}(\overline{\mathbf{C}}(f,-),\mathrm{F})(\alpha) &= y_{\mathrm{F},\mathbf{c}_2}\left(\alpha \circ \overline{\mathbf{C}}(f,-)\right) = \\ &= \left(\alpha \circ \overline{\mathbf{C}}(f,-)\right)_{\mathbf{c}_2}(\mathrm{id}_{\mathbf{c}_2}) = \alpha_{\mathbf{c}_2}(\mathrm{id}_{\mathbf{c}_2} \circ f) = \alpha_{\mathbf{c}_2}(f) \end{aligned} \tag{A.28}$$

and by Equation A.20

$$\alpha_{\mathbf{c}_2}(f) = \mathrm{F}(f)(u^{\alpha}). \tag{A.29}$$

✠

Lemma A.2.2. *Let* $\mathbf{C}$ *be a category,* $\mathrm{F}:\mathbf{C}\to\mathbf{Set}$ *a functor and* $*$ *a set with only one element. Then there is a natural isomorphism*

$$\varphi^{\mathrm{F}} : \overline{\mathbf{Set}}(*,\mathrm{F}-) \to \mathrm{F} \tag{A.30}$$

defined for each $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *and* $f : * \to \mathrm{F}(c)$ *by*

$$\varphi^{\mathrm{F}}_{\mathbf{c}}(f) = f(*). \tag{A.31}$$

Proof. Just a routine check. ✠

Definition A.2.9. Let $\mathbf{C}$ be a category, $\mathrm{F}:\mathbf{C}\to\mathbf{Set}$. A ***representation of*** F is a pair $(\mathbf{r},\psi)$ where $\mathbf{r} \in \mathcal{O}(\mathbf{C})$ and ψ is a natural isomorphism between $\overline{\mathbf{C}}(\mathbf{r},-)$ and F. The object $\mathbf{r}$ is called a ***representing object of*** F. A functor is said to be ***representable*** if a representation of its exists.

Lemma A.2.3. *The functors* $\overline{\mathbf{C}}(\mathbf{c}_1,-)$, $\overline{\mathbf{C}}(\mathbf{c}_2,-)$ *are naturally isomorphic if and only if the object* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *are isomorphic.*

Proof. If $i : \mathbf{c}_1 \to \mathbf{c}_2$ is an isomorphism, then it is easy to check that $\psi : \overline{\mathbf{C}}(\mathbf{c}_1,-) \to \overline{\mathbf{C}}(\mathbf{c}_2,-)$ defined for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ by $\psi_{\mathbf{c}} : \overline{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}) \to \overline{\mathbf{C}}(\mathbf{c}_2,\mathbf{c})$ as $\psi_{\mathbf{c}}(f) = fi^{-1}$ is a natural isomorphism.

Let $\psi : \overline{\mathbf{C}}(\mathbf{c}_1, -) \to \overline{\mathbf{C}}(\mathbf{c}_2, -)$ be a natural isomorphism. Set $i = \phi_{\mathbf{c}_1}(\mathrm{id}_{\mathbf{c}_1})$, $j = \phi^{-1}_{\mathbf{c}_2}(\mathrm{id}_{\mathbf{c}_2})$. From the commutative diagram

$$\begin{array}{ccc} \overline{\mathbf{C}}(\mathbf{c}_1\, \mathbf{c}_1) & \xrightarrow{\psi_{\mathbf{c}_1}} & \overline{\mathbf{C}}(\mathbf{c}_2, \mathbf{c}_1) \\ {\scriptstyle \overline{\mathbf{C}}(\mathbf{c}_1, j)}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(\mathbf{c}_2, j)} \\ \overline{\mathbf{C}}(\mathbf{c}_1\, \mathbf{c}_2) & \xrightarrow{\psi_{\mathbf{c}_2}} & \overline{\mathbf{C}}(\mathbf{c}_2, \mathbf{c}_2) \end{array} \tag{A.32}$$

we have

$$\psi_{\mathbf{c}_2}\left(\overline{\mathbf{C}}(\mathbf{c}_1, j)\,(\mathrm{id}_{\mathbf{c}_1})\right) = \psi_{\mathbf{c}_2}\,(j\,\mathrm{id}_{\mathbf{c}_1}) = \mathrm{id}_{\mathbf{c}_2} = \overline{\mathbf{C}}(\mathbf{c}_2, j)\,(\psi_{\mathbf{c}_1}\,(\mathrm{id}_{\mathbf{c}_1})) = ji \tag{A.33}$$

and from the commutative diagram

$$\begin{array}{ccc} \overline{\mathbf{C}}(\mathbf{c}_2\, \mathbf{c}_2) & \xrightarrow{\psi^{-1}_{\mathbf{c}_2}} & \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \\ {\scriptstyle \overline{\mathbf{C}}(\mathbf{c}_2, i)}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(\mathbf{c}_1, i)} \\ \overline{\mathbf{C}}(\mathbf{c}_2\, \mathbf{c}_1) & \xrightarrow{\psi^{-1}_{\mathbf{c}_1}} & \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_1) \end{array} \tag{A.34}$$

we have

$$\psi^{-1}_{\mathbf{c}_1}\left(\overline{\mathbf{C}}(\mathbf{c}_2, i)\,(\mathrm{id}_{\mathbf{c}_2})\right) = \psi^{-1}_{\mathbf{c}_1}\,(i\,\mathrm{id}_{\mathbf{c}_2}) = \mathrm{id}_{\mathbf{c}_1} = \overline{\mathbf{C}}(\mathbf{c}_1, i)\left(\psi^{-1}_{\mathbf{c}_2}\,(\mathrm{id}_{\mathbf{c}_2})\right) = ij. \tag{A.35}$$

✠

Proposition A.2.13. *Let* $\mathrm{F}:\mathbf{C}{\to}\mathbf{Set}$ *and* $*$ *a set with one element. If* $(\mathbf{r}, u)$ *is a universal arrow from* $*$ *to* F *then there is a representation* $(\mathbf{r}, \psi)$ *of* F *defined for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *and* $f : \mathbf{r} \to \mathrm{F}(\mathbf{c})$ *by* $\psi_{\mathbf{c}}(f) = \mathrm{F}(f)(u(*))$.

Proof. By Proposition A.2.9 if $(\mathbf{r}, u)$ is a universal arrow from $*$ to F then there is a natural isomorphism $\psi : \overline{\mathbf{C}}(\mathbf{r}, -) \to \overline{\mathbf{Set}}(*, \mathrm{F}-)$ such that for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ and $f : \mathbf{r} \to \mathbf{c}$ $\psi_{\mathbf{c}}(f) = \mathrm{F}(f)u$. The thesis then follows from Lemma A.2.2. ✠

Proposition A.2.14. *Let* $\mathrm{F}:\mathbf{C}{\to}\mathbf{Set}$. *Then* $\psi : \overline{\mathbf{C}}(\mathbf{r}, -) \to \mathrm{F}$ *is a representation of* F *if and only if* $(\mathbf{r}, u^{\psi})$ *is an initial object of* $\mathbf{C}[\mathrm{F}]$.

Proof. Let $\psi : \overline{\mathbf{C}}(\mathbf{r}, -) \to \mathrm{F}$ be a representation of F. If $(\mathbf{c}, v) \in \mathcal{O}(\mathbf{C}[\mathrm{F}])$ then $v \in \mathrm{F}(\mathbf{c})$ so let $f = \psi^{-1}_{\mathbf{c}}(v)$. Since the diagram

$$\begin{array}{ccc} \overline{\mathbf{C}}(\mathbf{r}, \mathbf{r}) & \xrightarrow{\psi_{\mathbf{r}}} & \mathrm{F}(\mathbf{r}) \\ {\scriptstyle \overline{\mathbf{C}}(\mathbf{r}, f)}\downarrow & & \downarrow{\scriptstyle \mathrm{F}(f)} \\ \overline{\mathbf{C}}(\mathbf{r}, \mathbf{c}) & \xrightarrow{\psi_{\mathbf{c}}} & \mathrm{F}(\mathbf{c}) \end{array} \tag{A.36}$$

commutes we have

$$\mathrm{F}(f) \circ \psi_{\mathbf{r}}(\mathrm{id}_{\mathbf{r}}) = \mathrm{F}(f)(u^{\psi}) = \psi_{\mathbf{c}} \circ \overline{\mathbf{C}}(\mathbf{r}, f)(\mathrm{id}_{\mathbf{r}}) = \psi_{\mathbf{c}}(f) = v \tag{A.37}$$

so $f \in \overline{\mathbf{C}[\mathrm{F}]}((\mathbf{r}, u^{\psi}), (\mathbf{c}, v))$. If $g \in \overline{\mathbf{C}[\mathrm{F}]}((\mathbf{r}, u^{\psi}), (\mathbf{c}, v))$ then $v = \mathrm{F}(g)(u^{\psi})$ and $\psi_{\mathbf{c}}(g) = \mathrm{F}(g)(u^{\psi}) = v$, so $g = f$ since $\psi_{\mathbf{c}}$ is a bijection. Thus $(\mathbf{r}, u^{\psi})$ is an initial object of $(\mathbf{C}[\mathrm{F}])$. Let $(\mathbf{r}, u^{\psi})$ be an initial object of $(\mathbf{C}[\mathrm{F}])$, let's show that ψ is a natural isomorphism. If $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ and $v \in \mathrm{F}(\mathbf{c})$ there is exactly one morphism $f : (\mathbf{r}, u^{\psi}) \to (\mathbf{c}, v)$, and $v = \mathrm{F}(f)u^{\psi}$ so $v = \psi_{\mathbf{c}}(f)$. ✠

Definition A.2.10. Let $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ be a functor, $\mathbf{d} \in \mathcal{O}(\mathbf{D})$. An ***arrow from*** F ***to*** $\mathbf{d}$ is a pair $(\mathbf{c}, u)$ where $\mathbf{c} \in \mathcal{O}(\mathbf{C})$, $u \in \overline{\mathbf{D}}(\mathrm{F}(c), \mathbf{d})$.

Definition A.2.11. Let $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ be a functor, $\mathbf{d} \in \mathcal{O}(\mathbf{D})$. Two arrows $(\mathbf{c}_1, u_1)$, $(\mathbf{c}_2, u_2)$ from F to $\mathbf{d}$ are ***isomorphic*** if there is an isomorphism $i : c_1 \to c_2$ such that $u_1 = u_2 \circ \mathrm{F}(i)$.

Definition A.2.12. Let $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ be a functor, $\mathbf{d} \in \mathcal{O}(\mathbf{D})$. An arrow $(\mathbf{c}, u)$ from F to $\mathbf{d}$ is ***universal*** if and only if for any arrow $(\mathbf{c}_*, u_*)$ from F to $\mathbf{d}$ there is a unique morphism $f : \mathbf{c}_* \to \mathbf{c}$ such that $u_* = u \circ \mathrm{F}(f)$, that is, the diagram

$$\begin{array}{ccc} F(\mathbf{c}) & \xrightarrow{u} & \mathbf{d} \\ {\scriptstyle F(f)}\uparrow & \nearrow_{u_*} & \\ F(\mathbf{c}_*) & & \end{array} \tag{A.38}$$

commutes.

Proposition A.2.15. *Any two universal arrows from a functor* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ *to an object* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$ *are isomorphic.*

Proof. Let $(\mathbf{c}_1, u_1)$ and $(\mathbf{c}_2, u_2)$ both be universal arrows from to F $\mathbf{d}$. Then, since $(\mathbf{c}_1, u_1)$ is universal there is $f : \mathbf{c}_2 \to \mathbf{c}_1$ such that $u_1 = F(f) \circ u_2$, and since $(\mathbf{c}_2, u_2)$ is universal there is $g : \mathbf{c}_1 \to \mathbf{c}_2$ such that $u_2 = F(g) \circ u_1$. So $u_1 = F(f) \circ F(g) \circ u_1 = F(f \circ g) \circ u_1$ whence $f \circ g = \mathrm{id}_{\mathbf{c}_1}$, because $(\mathbf{c}_1, u_1)$ is universal. Also $u_2 = F(g) \circ F(f) \circ u_2 = F(g \circ f) \circ u_2$ whence $g \circ f = \mathrm{id}_{\mathbf{c}_2}$, because $(\mathbf{c}_2, u_2)$ is universal. So g is an isomorphism. ✠

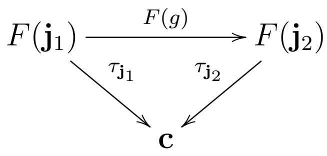
ADD HERE PROPOSITIONS AND COROLLARIES AS FOR UNIVERSAL ARROWS FROM OBJECTS TO FUNCTORS

A.2.1 Direct limits

Definition A.2.13. Let $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$ be a functor. A ***direct target for*** F is an arrow from F to the diagonal functor Δ_{J}^{C}.

Remark A.2.2. According to this definition, a direct target for the functor $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$ is a pair $(\mathbf{c}, \tau)$ where $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ and τ is a natural transformation $\tau : \mathrm{F} \to \mathrm{C}_{\mathbf{c}}^{\mathbf{J},\mathbf{C}}$. That is, for

$\mathbf{j} \in \mathcal{O}(\mathbf{J})$, $\tau_{\mathbf{j}} : F(\mathbf{j}) \to \mathbf{c}$, and for any $\mathbf{j}_1 \in \mathcal{O}(\mathbf{J})$ and $\mathbf{j}_2 \in \mathcal{O}(\mathbf{J})$ and any $g : \mathbf{j}_1 \to \mathbf{j}_2$ the diagram



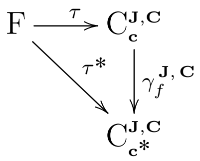
(A.39)

commutes.

Definition A.2.14. Let $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$ be a functor. A ***direct limit for*** F is a universal arrow from F to the diagonal functor Δ^{C}_{J}.

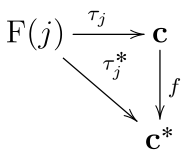
Definition A.2.15. If $(\mathbf{c}, \tau)$ is a direct limit for the functor F, then $\mathbf{c}$ is a ***direct limit object*** for F and τ is a ***direct limit cone*** for F.

Remark A.2.3. According to Definition A.2.15, a direct limit for the functor $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$ is a direct target $(\mathbf{c}, \tau)$ for F such that if $(\mathbf{c}^*, \tau^*)$ is any direct target for F, there is a unique morphism $f : \mathbf{c} \to \mathbf{c}^*$ such that $\tau^* = \gamma_f^{\mathbf{J},\mathbf{C}} \circ \tau$, that is the diagram



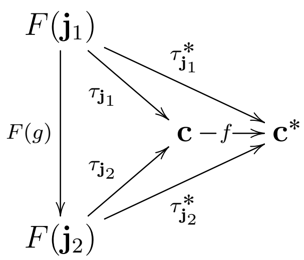
(A.40)

commutes, that is, for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ the diagram



(A.41)

Commutes. More in details, for any $\mathbf{j}_1 \in \mathcal{O}(\mathbf{J})$ and $\mathbf{j}_2 \in \mathcal{O}(\mathbf{J})$ and any $g : \mathbf{j}_1 \to \mathbf{j}_2$ the diagram



(A.42)

commutes.

Proposition A.2.16. *Let* $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$ *be a functor,* $\mathbf{d} \in \mathcal{O}(\mathbf{C})$*. An inverse target* $(\mathbf{l}, \lambda)$ *to* F *is a limit of* F *if and only if there is a natural bijection* $\phi : \overline{\mathbf{C}}\,(\mathbf{l}, -) \to \overline{\mathbf{C}^{\mathbf{J}}}\left(\mathrm{F}, \Delta^{\mathbf{J}}_{\mathbf{C}}-\right)$ *and* $\lambda = \phi_{\mathbf{l}}(\mathrm{id}_{\mathbf{l}})$.

Proof. Straightforward from Corollary A.2.1. ✠

Proposition A.2.17. *Any two direct limits for a functor are isomorphic.*

Proof. A direct consequence of Proposition A.2.12 ✠

Notation A.2.3. We will write

$$\varinjlim \mathrm{F} \tag{A.43}$$

for the isomorphism class of the direct limit of the functor F.

Proposition A.2.18. *Let* $\mathrm{F}:\mathbf{J}\to\mathbf{C}$, $\mathrm{G}:\mathbf{J}\to\mathbf{C}$ *be functors,* $\tau : \mathrm{F} \to \mathrm{G}$, $(\mathbf{f}, \phi) \in \varinjlim \mathrm{F}$, $(\mathbf{g}, \psi) \in \varinjlim \mathrm{G}$. *Then there is a unique morphism* $h : \mathbf{f} \to \mathbf{g}$ *such that* $\psi \circ \tau = \gamma_h^{\mathbf{J},\mathbf{C}} \circ \phi$, *that is, the diagram*

$$\begin{array}{ccc} \mathrm{F} & \xrightarrow{\tau} & \mathrm{G} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathrm{C}_{\mathbf{f}}^{\mathbf{J},\mathbf{C}} & \xrightarrow[\gamma_h^{\mathbf{J},\mathbf{C}}]{} & \mathrm{C}_{\mathbf{f}}^{\mathbf{J},\mathbf{C}} \end{array} \tag{A.44}$$

commutes, or, more in details, for each $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\psi_{\mathbf{j}} \circ \tau_{\mathbf{j}} = h \circ \phi_{\mathbf{j}}$, *that is, for each* $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ *the diagram*

$$\begin{array}{ccc} F(\mathbf{j}) & \xrightarrow{\tau_{\mathbf{j}}} & G(\mathbf{j}) \\ {\scriptstyle \phi_{\mathbf{j}}}\downarrow & & \downarrow{\scriptstyle \psi_{\mathbf{j}}} \\ \mathbf{f} & \xrightarrow[h]{} & \mathbf{g} \end{array} \tag{A.45}$$

commutes.

Proof. Let $\psi^* = \psi \circ \tau$. Then $(\mathbf{g}, \psi^*)$ is a direct target for F, so there is a unique morphism $h : \mathbf{f} \to \mathbf{g}$ such that $\psi^* = \gamma_h^{\mathbf{J},\mathbf{C}} \circ \phi$, that is, $\psi \circ \tau = \gamma_h^{\mathbf{J},\mathbf{C}} \circ \phi$. ✠

Definition A.2.16. Let $\mathrm{F}:\mathbf{J}\to\mathbf{C}$, $\mathrm{g}:\mathbf{J}\to\mathbf{C}$ be functors, $\tau : \mathrm{F} \to \mathrm{G}$, $(\mathbf{f}, \phi) \in \varinjlim \mathrm{F}$, $(\mathbf{g}, \psi) \in \varinjlim \mathrm{G}$, $h : \mathbf{f} \to \mathbf{g}$ such that $\psi \circ \tau = \gamma_h^{\mathbf{J},\mathbf{C}} \circ \phi$. Then h is called a ***direct limit*** for the natural transformation τ.

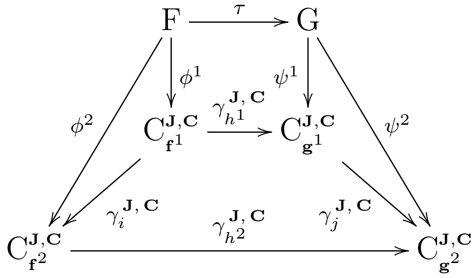
Notation A.2.4. Let $\mathrm{F}:\mathbf{J}\to\mathbf{C}$, $\mathrm{G}:\mathbf{J}\to\mathbf{C}$ be functors, $\tau : \mathrm{F} \to \mathrm{G}$. We will write

$$\varinjlim\, [\tau, (f, \phi), (g, \psi)] \tag{A.46}$$

for the direct limit of τ relative to the direct limits $(\mathbf{f}, \phi)$ of F and $(\mathbf{g}, \psi)$ of G.

Proposition A.2.19. *Let* $\mathrm{F}:\mathbf{J}\to\mathbf{C}$, $\mathrm{G}:\mathbf{J}\to\mathbf{C}$, $\tau : \mathrm{F} \to \mathrm{G}$ *a natural transformation,* $h^1 = \varinjlim\, [\tau, (\mathbf{f}^1, \phi^1), (\mathbf{g}^1, \psi^1)]$, $h^2 = \varinjlim\, [\tau, (\mathbf{f}^2, \phi^2), (\mathbf{g}^2, \psi^2)]$. *If* $i : \mathbf{f}^1 \to \mathbf{f}^2$ *and* $j : \mathbf{g}^1 \to \mathbf{g}^2$ *are isomorphisms such that* $\phi^2 = \gamma_i^{\mathbf{J},\mathbf{C}} \circ \phi^1$ *and* $\psi^2 = \gamma_j^{\mathbf{J},\mathbf{C}} \circ \psi^1$, *then* $j \circ h^1 = h^2 \circ i$. *In particular, any two direct limits of* τ *are equivalent.*

Proof. We have $\psi^2 \circ \tau = \gamma_j^{\mathbf{J},\mathbf{C}} \circ \psi^1 \circ \tau = \gamma_j^{\mathbf{J},\mathbf{C}} \circ \gamma_{h^1}^{\mathbf{J},\mathbf{C}} \circ \phi^1 = \gamma_{j \circ h^1}^{\mathbf{J},\mathbf{C}} \circ \phi^1 = \gamma_{h^2}^{\mathbf{J},\mathbf{C}} \circ \phi^2 = \gamma_{h^2}^{\mathbf{J},\mathbf{C}} \circ \gamma_i^{\mathbf{J},\mathbf{C}} \circ \phi^1 = \gamma_{h^2 \circ i}^{\mathbf{J},\mathbf{C}} \circ \phi^1$:



Since $(\mathbf{g}^2, \psi^2 \circ \tau)$ is a direct target for F, there is only one morphism $k : \mathbf{f}^1 \to \mathbf{g}^2$ such that $\gamma_k^{\mathbf{J},\mathbf{C}} \circ \phi^1 = \psi^2 \circ \tau$, so $j \circ h^1 = h^2 \circ i$ follows. ✠

Proposition A.2.20. *Let* $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$, $\mathrm{G}{:}\mathbf{J}{\to}\mathbf{C}$ *be functors,* $\tau : \mathrm{F} \to \mathrm{G}$, $\sigma : \mathrm{F} \to \mathrm{G}$, *and suppose*

$$\varinjlim [\tau, (\mathbf{f}, \phi), (\mathbf{g}, \psi)] = \varinjlim [\sigma, (\mathbf{f}, \phi), (\mathbf{g}, \psi)]$$

for certain direct limits $(\mathbf{f}, \phi)$ *of* F *and* $(\mathbf{g}, \gamma)$ *of* G*. Then*

$$\varinjlim [\tau, (\mathbf{f}', \phi'), (\mathbf{g}', \psi')] = \varinjlim [\sigma, (\mathbf{f}', \phi'), (\mathbf{g}', \psi')]$$

for any direct limits $(\mathbf{f}', \phi')$ *of* F *and* $(\mathbf{g}', \psi')$ *of* G.

Proof. Let

$$\begin{aligned} h &= \varinjlim [\tau, (\mathbf{f}, \phi), (\mathbf{g}, \psi)] = \varinjlim [\sigma, (\mathbf{f}, \phi), (\mathbf{g}, \psi)] \\ t &= \varinjlim [\tau, (\mathbf{f}', \phi'), (\mathbf{g}', \psi')] \\ s &= \varinjlim [\sigma, (\mathbf{f}', \phi'), (\mathbf{g}', \psi')]; \end{aligned}$$

If $i : \mathbf{f} \to \mathbf{f}'$ and $j : \mathbf{g} \to \mathbf{g}'$ are isomorphisms such that $\phi' = \gamma_i^{\mathbf{J},\mathbf{C}} \circ \phi$ and $\psi' = \gamma_j^{\mathbf{J},\mathbf{C}} \circ \psi$, then $j \circ h = t \circ i$ and $j \circ h = s \circ i$, whence $t = s$. ✠

Notation A.2.5. Let $\mathrm{F}{:}\mathbf{J}{\to}\mathbf{C}$, $\mathrm{G}{:}\mathbf{J}{\to}\mathbf{C}$ be functors, $\tau : \mathrm{F} \to \mathrm{G}$. We will write

$$\varinjlim \tau \tag{A.47}$$

for the equivalence class of direct limits of τ.

Proposition A.2.21. *If every functor from* **J** *to* **C** *has a direct limit, there is a functor*

$$\varinjlim{:}\mathbf{C}^{\mathbf{J}}{\to}\mathbf{C}^{\,\mathrm{q}}.$$

Proof. A direct consequence of Propositions A.2.18 and A.2.19. ✠

Proposition A.2.22. *Let* $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ *be a functor. If* $\mathbf{J}$ *has a terminal object* $\mathbf{t}$*, for* $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ *let* $f_{\mathbf{j}}$ *be the unique element of* $\overline{\mathbf{J}}(\mathbf{j},\mathbf{t})$*,* $\mathbf{t}^* = \mathrm{F}(\mathbf{t})$*,* $\tau : \mathrm{F} \to \mathrm{C}^{\mathbf{J},\mathbf{C}}_{\mathbf{t}^*}$ *the natural transformation defined, for* $\mathbf{j} \in \mathcal{O}(\mathbf{J})$*, by* $\tau_{\mathbf{j}} = \mathrm{F}(f_{\mathbf{j}})$*. Then* $(\mathbf{t}^*, \tau)$ *is a direct limit for* F.

Proof. Of course $(\mathbf{t}^*, \tau)$ is a direct target for F. Let $(\mathbf{s}, \sigma)$ be a direct target for F. Then $\sigma_{\mathbf{t}} : \mathbf{t}^* \to \mathbf{s}$ and for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ we have $\sigma_{\mathbf{j}} = \sigma_{\mathbf{t}} \circ \tau_{\mathbf{j}}$. If $g : \mathbf{t}^* \to \mathbf{s}$ is such that for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ we have $\sigma_{\mathbf{j}} = g \circ \tau_{\mathbf{j}}$, then in particular $\sigma_{\mathbf{t}} = g \circ \tau_{\mathbf{t}}$, but $\tau_{\mathbf{t}} = F(f_{\mathbf{t}}) = F(\mathrm{id}_{\mathbf{t}}) = \mathrm{id}_{\mathbf{t}^*}$, so $g = \sigma_{\mathbf{t}}$. ✠

Definition A.2.17. Let $\mathrm{F}:\mathbf{A}\to\mathbf{B}$, $\mathrm{G}:\mathbf{B}\to\mathbf{C}$ be functors. We say that G ***creates direct limits for*** F if for every direct limit $(\mathbf{c}, \tau)$ of $\mathrm{G} \circ \mathrm{F}$ there exists a unique direct limit $(\mathbf{b}, \sigma)$ of F such that

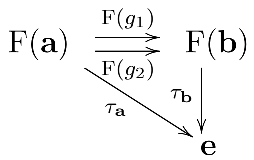
- $\mathrm{G}(\mathbf{b}) = \mathbf{c}$
- $\forall \mathbf{a} \in \mathcal{O}(\mathbf{A})\ \tau_{\mathbf{a}} = \mathrm{G}(\sigma_{\mathbf{a}})$.

Definition A.2.18. A ***direct equalizer*** in a category $\mathbf{C}$ is a direct limit for a functor $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ where $\mathbf{J}$ is a category of type $\{\cdot \rightrightarrows \cdot\}$.

Remark A.2.4. Let $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ where

1. $\mathcal{O}(\mathbf{J}) = \{\mathbf{a}, \mathbf{b}\}$
2. $\overline{\mathbf{J}}(\mathbf{a}, \mathbf{b}) = \{g_1, g_2\}$

and let $(\mathbf{e}, \tau)$ be a direct equalizer for F. Then the following diagram commutes



that is, $\tau_{\mathbf{a}} = \tau_{\mathbf{b}} \circ \mathrm{F}(g_1) = \tau_{\mathbf{b}} \circ \mathrm{F}(g_2)$, and if $(\mathbf{f}, \sigma)$ is a direct target for F, which means that $\sigma_{\mathbf{a}} = \sigma_{\mathbf{b}} \circ \mathrm{F}(g_1) = \sigma_{\mathbf{b}} \circ \mathrm{F}(g_2)$, then there is a unique morphism $h : \mathbf{e} \to \mathbf{f}$ such that $\sigma_{\mathbf{a}} = h \circ \tau_{\mathbf{a}}$ and $\sigma_{\mathbf{b}} = h \circ \tau_{\mathbf{b}}$. So we can restate the definition of direct equalizer in the following terms, that refer to a pair of parallel morphisms rather than to a functor. The morphism $i : \mathbf{d} \to \mathbf{e}$ is a ditect equalizer for the pair of parallel morphisms $f_1 : \mathbf{c} \to \mathbf{d}$ and $f_2 : \mathbf{c} \to \mathbf{d}$ if

- $i \circ f_1 = i \circ f_2$
- for any morphism $j : \mathbf{d} \to \mathbf{f}$ such that $j \circ f_1 = j \circ f_2$ there is a unique morphism $h : \mathbf{e} \to \mathbf{f}$ such that $j = h \circ i$.

Proposition A.2.23. *If* $i : \mathbf{b} \to \mathbf{e}$ *is a direct equalizer then* i *is epi.*

Proof. Let $i : \mathbf{b} \to \mathbf{e}$ be a direct equalizer for the pair $f_1 : \mathbf{a} \to \mathbf{b}$ and $f_2 : \mathbf{a} \to \mathbf{b}$. Let $g_1 : \mathbf{e} \to \mathbf{f}$ and $g_2 : \mathbf{e} \to \mathbf{f}$ be such that $g_1 \circ i = g_2 \circ i$. Let $h = g_1 \circ i = g_2 \circ i$. Then $h \circ f_1 = h \circ f_2$, so there's a unique $g : \mathbf{e} \to \mathbf{f}$ such that $h = g \circ i$. Then $g_1 \circ i = g_2 \circ i$ implies $g_1 = g_2$. ✠

Lemma A.2.4. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$, $\mathrm{G}{:}\mathbf{C}{\to}\mathbf{D}$, $\mathrm{f} : \mathrm{F} \to \mathrm{G}$, $\mathrm{g} : \mathrm{F} \to \mathrm{G}$ *and for each* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *let* $h_{\mathbf{c}}$ *be a direct equalizer for the pair* $\mathrm{f}_{\mathbf{c}}, \mathrm{g}_{\mathbf{c}}$. *Then there exists a functor* $\mathrm{E}^h{:}\mathbf{C}{\to}\mathbf{D}$ *such that for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\mathrm{E}^h(\mathbf{c}) = \mathrm{cod}(h_{\mathbf{c}})$ *and for* $k \in \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ $\mathrm{E}^h(k)$ *is the unique morphism such that the diagram*

$$\begin{array}{ccc} \mathrm{G}(\mathbf{c}_1) & \xrightarrow{h_{\mathbf{c}_1}} & \mathrm{E}^h(\mathbf{c}_1) \\ \downarrow{\scriptstyle \mathrm{G}(k)} & & \downarrow{\scriptstyle \mathrm{E}^h(k)} \\ \mathrm{G}(\mathbf{c}_2) & \xrightarrow{h_{\mathbf{c}_2}} & \mathrm{E}^h(\mathbf{c}_2) \end{array}$$

commutes, and a natural transformation $\mathrm{e}^h : \mathrm{D} \to \mathrm{E}^h$ *such that for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\mathrm{e}^h_{\mathbf{c}} = h_{\mathbf{c}}$.

Proof. We have

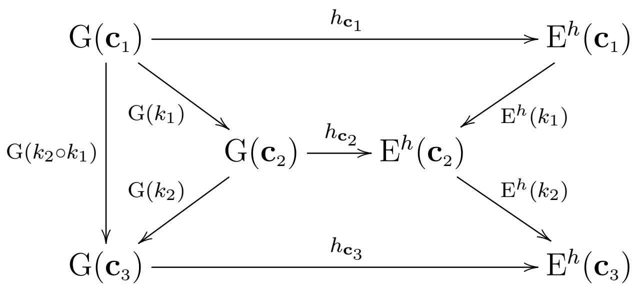
$$h_{\mathbf{c}_2} \circ \mathrm{G}(k) \circ \mathrm{f}_{\mathbf{c}_1} = h_{\mathbf{c}_2} \circ \mathrm{f}_{\mathbf{c}_2} \circ \mathrm{F}(k) = h_{\mathbf{c}_2} \circ \mathrm{g}_{\mathbf{c}_2} \circ \mathrm{F}(k) = h_{\mathbf{c}_2} \circ \mathrm{G}(k) \circ \mathrm{g}_{\mathbf{c}_1}$$

so there is a unique morphism $\mathrm{E}^h(k) : \mathrm{E}^h(\mathbf{c}_1) \to \mathrm{E}^h(\mathbf{c}_2)$ such that $h_{\mathbf{c}_2} \circ \mathrm{G}(k) = \mathrm{E}^h(k) \circ h_{\mathbf{c}_1}$. Since the diagram

$$\begin{array}{ccc} \mathrm{G}(\mathbf{c}) & \xrightarrow{h_{\mathbf{c}}} & \mathrm{E}^h(\mathbf{c}) \\ {\scriptstyle \mathrm{id}_{\mathrm{G}(\mathbf{c})}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{E}^h(\mathbf{c})}} \\ \mathrm{G}(\mathbf{c}) & \xrightarrow{h_{\mathbf{c}}} & \mathrm{E}^h(\mathbf{c}) \end{array}$$

commutes, then $\mathrm{E}^h(\mathrm{id}_{\mathbf{c}}) = \mathrm{id}_{\mathrm{E}^h(\mathbf{c})}$.

The diagram



commutes, so

$$h_{\mathbf{c}_3} \circ \mathrm{G}(k_2 \circ k_1) = h_{\mathbf{c}_3} \circ \mathrm{G}(k_2) \circ \mathrm{G}(k_1) = \mathrm{E}^h(k_2) \circ h_{\mathbf{c}_2} \circ \mathrm{G}(k_1) = \mathrm{E}^h(k_2) \circ \mathrm{E}^h(k_1) \circ h_{\mathbf{c}_1}$$

that is, $\mathrm{E}^h(k_2 \circ k_1) = \mathrm{E}^h(k_2) \circ \mathrm{E}^h(k_1)$.

That the morphisms $h_{\mathbf{c}}$ define a natural transformation follows immediately. ✠

Proposition A.2.24. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathrm{F}{:}\mathbf{C}\to\mathbf{D}$, $\mathrm{G}{:}\mathbf{C}\to\mathbf{D}$, $\mathrm{f}:\mathrm{F}\to\mathrm{G}$, $\mathrm{g}:\mathrm{F}\to\mathrm{G}$ *and for each* $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ *let* $h_{\mathbf{c}}$ *be a direct equalizer for the pair* $\mathrm{f}_{\mathbf{c}},\mathrm{g}_{\mathbf{c}}$. *Then the natural transformation* e^h *is a direct equalizer for the pair* f,g.

Proof. If $\mathrm{k}:\mathrm{G}\to\mathrm{K}$ is such that $\mathrm{k}\circ\mathrm{f}=\mathrm{k}\circ\mathrm{g}$, then, for each $\mathbf{c}\in\mathcal{O}(\mathbf{C})$, $\mathrm{k}_{\mathbf{c}}\circ\mathrm{f}_{\mathbf{c}}=\mathrm{k}_{\mathbf{c}}\circ\mathrm{g}_{\mathbf{c}}$, so there is a unique $i_{\mathbf{c}}:\mathrm{H}(\mathbf{c})\to\mathrm{K}(\mathbf{c})$ such that $\mathrm{k}_{\mathbf{c}}=\mathrm{i}_{\mathbf{c}}\circ h_{\mathbf{c}}$. The morphisms $i_{\mathbf{c}}$ define a natural transformation from H to K since, for a morphism $j\in\overline{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)$, the diagram

$$\begin{array}{ccccc}
\mathrm{G}(\mathbf{c}_1) & \xrightarrow{\mathrm{h}_{\mathbf{c}_1}} & \mathrm{H}(\mathbf{c}_1) & \xrightarrow{i_{\mathbf{c}_1}} & \mathrm{K}(\mathbf{c}_1)\\
{\scriptstyle \mathrm{G}(j)}\downarrow & & {\scriptstyle \mathrm{H}(j)}\downarrow & & \downarrow{\scriptstyle \mathrm{K}(j)}\\
\mathrm{G}(\mathbf{c}_2) & \xrightarrow{\mathrm{h}_{\mathbf{c}_2}} & \mathrm{H}(\mathbf{c}_2) & \xrightarrow{i_{\mathbf{c}_2}} & \mathrm{K}(\mathbf{c}_2)
\end{array}$$

commutes because

$$i_{\mathbf{c}_2}\circ\mathrm{H}(j)\circ\mathrm{h}_{\mathbf{c}_1}=i_{\mathbf{c}_2}\circ\mathrm{h}_{\mathbf{c}_2}\circ\mathrm{G}(j)=\mathrm{k}_{\mathbf{c}_2}\circ\mathrm{G}(j)=\mathrm{K}(j)\circ\mathrm{k}_{\mathbf{c}_1}=\mathrm{K}(j)\circ i_{\mathbf{c}_1}\circ\mathrm{h}_{\mathbf{c}_1}$$

whence, since $\mathrm{h}_{\mathbf{c}_1}$ is an epimorphism, $i_{\mathbf{c}_2}\circ\mathrm{H}(j)=\mathrm{K}(j)\circ i_{\mathbf{c}_1}$. ✠

Lemma A.2.5. *Let* τ *be an algebraic type,* $\mathbf{a}\in\mathcal{O}(\mathbf{Alg}_\tau)$ *and* $\mathbf{e}$ *an equivalence relation on* $\mathbf{a}$. *Then the following conditions are equivalent*

1. *The set* $\mathbf{a}/\mathbf{e}$ *can be given an algebraic structure in such a way that the map*

   $$\begin{aligned} p:\mathbf{a}&\to\mathbf{a}/\mathbf{e}\\ x&\mapsto[x]_{\mathbf{e}}\end{aligned}$$

   is a morphism in $\mathbf{Alg}_\tau$

2. *There are* $\mathbf{b}\in\mathcal{O}(\mathbf{Alg}_\tau)$ *and* $f:\mathbf{a}\to\mathbf{b}$ *such that* $\mathbf{e}=\{(x,y)\in\mathbf{a}\times\mathbf{a}\,|\,f(x)=f(y)\}$

3. $\mathbf{e}$ *is a subalgebra of* $\mathbf{a}\times\mathbf{a}$.

Proof.
Let's prove that 1. ⇒ 2. The object $\mathbf{a}/\mathbf{e}$ and the map p in 1. satisfy the conditions stated in 2. for $\mathbf{b}$ and f.
Let's prove that 2. ⇒ 3. Let $\omega\in\tau_o$, and $(x_i,y_i)\in\mathbf{e}$ for $i=1\ldots n_\omega$. Then

$$\begin{aligned} f\left(\omega_{\mathbf{a}}(x_1,\ldots,x_{n_\omega})\right)&=\omega_{\mathbf{a}}(f(x_1),\ldots,f(x_{n_\omega}))=\\ &=\omega_{\mathbf{a}}(f(y_1),\ldots,f(y_{n_\omega}))=\\ &=f\left(\omega_{\mathbf{a}}(y_1,\ldots,y_{n_\omega})\right)\end{aligned}$$

that is $(\omega_{\mathbf{a}}(x_1,\ldots,x_{n_\omega}),\omega_{\mathbf{a}}(y_1,\ldots,y_{n_\omega}))\in\mathbf{e}$, but

$$\omega_{\mathbf{a}\times\mathbf{a}}\left((x_1,y_1),\ldots,(x_{n_\omega},y_{n_\omega})\right)=(\omega_{\mathbf{a}}(x_1,\ldots,x_{n_\omega}),\omega_{\mathbf{a}}(y_1,\ldots,y_{n_\omega}))$$

thus $\omega_{\mathbf{a}\times\mathbf{a}}\left((x_1,y_1),\ldots,(x_{n_\omega},y_{n_\omega})\right)\in\mathbf{e}$.
Let's prove that 3. ⇒ 1. Let $\omega\in\tau_o$ and $x_i,y_i\in\mathbf{a}$ for $i=\ \ldots,n_\omega$ such that $[x_i]=[y_i]$ for $i=\ \ldots,n_\omega$. Then $(x_i,y_i)\in\mathbf{e}$ for $i=\ \ldots,n_\omega$, and $\omega_{\mathbf{a}\times\mathbf{a}}\left((x_1,y_1),\ldots,(x_{n_\omega},y_{n_\omega})\right)\in\mathbf{e}$, that is $(\omega_{\mathbf{a}}(x_1,\ldots,x_{n_\omega}),\omega_{\mathbf{a}}(y_1,\ldots,y_{n_\omega}))\in\mathbf{e}$ or $[\omega_{\mathbf{a}}(x_1,\ldots,x_{n_\omega})]=[\omega_{\mathbf{a}}(y_1,\ldots,y_{n_\omega})]$. This allows to define $\omega_{\mathbf{a}/\mathbf{e}}([x_1],\ldots,[x_{n_\omega}])=[\omega_{\mathbf{a}}(x_1,\ldots,x_{n_\omega})]$. ✠

Definition A.2.19. Let τ be an algebraic type, $\mathbf{a}\in\mathcal{O}(\mathbf{Alg}_\tau)$. A ***congruence on*** $\mathbf{a}$ is an equivalence relation on $\mathbf{a}$ that is a subalgebra of $\mathbf{a}\times\mathbf{a}$.

Lemma A.2.6. *Let τ be an algebraic type, $\mathbf{a}\in\mathcal{O}(\mathbf{Alg}_\tau)$. The intersection of any family of congruences on $\mathbf{a}$ is a congruence on $\mathbf{a}$.*

Proof. Routine check. ✠

Definition A.2.20. Let τ be an algebraic type, $\mathbf{a}\in\mathcal{O}(\mathbf{Alg}_\tau)$, $\mathbf{r}\subseteq\mathbf{r}\times\mathbf{r}$. The intersection of all the congruences on $\mathbf{a}$ that contains $\mathbf{r}$ is the ***congruence generated by*** $\mathbf{r}$ and will be denoted by $\langle\mathbf{r}\rangle$.

Proposition A.2.25. *Let τ be an algebraic type, $\mathbf{a}\in\mathcal{O}(\mathbf{Alg}_\tau)$, $\mathbf{r}\subseteq\mathbf{a}\times\mathbf{a}$, $p_{\mathbf{a}}$ the map*

$$\begin{aligned} p_{\mathbf{a}}:\mathbf{a}&\to\mathbf{a}/\langle\mathbf{r}\rangle\\ x&\mapsto[x]_{\langle\mathbf{r}\rangle}.\end{aligned}$$

Then for any morphism $f:\mathbf{a}\to\mathbf{b}$ such that $f(x)=f(y)$ for each $(x,y)\in\mathbf{r}$ there is a unique morphism $f^*:\mathbf{a}/\langle\mathbf{r}\rangle\to\mathbf{b}$ such that $f=f^*\circ p_{\mathbf{a}}$.

Proof. The set

$$\mathbf{s}=\{(x,y)\in\mathbf{a}\times\mathbf{a}\,|\,f(x)=f(y)\}\,.$$

is a congruence on $\mathbf{a}$, thus $\langle\mathbf{r}\rangle\subseteq\mathbf{s}$. If for $x,y\in\mathbf{a}$ $[x]_{\langle\mathbf{r}\rangle}=[y]_{\langle\mathbf{r}\rangle}$, then $(x,y)\in\langle\mathbf{r}\rangle$ whence $(x,y)\in\mathbf{s}$, thus $f(x)=f(y)$. This means that it is possible to define

$$\begin{aligned} f^*:\mathbf{a}/\langle\mathbf{r}\rangle&\to\mathbf{b}\\ [x]_{\langle\mathbf{r}\rangle}&\mapsto f(x)\end{aligned}$$

and $f=f^*\circ p_{\mathbf{a}}$.
If $g:\mathbf{s}/\langle\mathbf{r}\rangle\to\mathbf{b}$ is such that $f=g\circ p_{\mathbf{a}}$, let $u\in\mathbf{a}/\langle\mathbf{r}\rangle$; then $u=[x]_{\langle\mathbf{r}\rangle}$ for some $x\in\mathbf{a}$, so $g(u)=g\big([x]_{\langle\mathbf{r}\rangle}\big)=f(x)=f^*\big([x]_{\langle\mathbf{r}\rangle}\big)=f^*(u)$. ✠

Notation A.2.6. For a pair $f_1:\mathbf{a}\to\mathbf{b}$, $f_2:\mathbf{a}\to\mathbf{b}$ of parallel morphisms in $\mathbf{Alg}_\tau$ set

$$r_0(f_1,f_2)=\{(x,y)\in\mathbf{b}\times\mathbf{b}\,|\,\exists z\in\mathbf{a}:x=f(z)\wedge y=g(z)\}$$

and

$$\mathbf{r}(f_1,f_2)=\langle r_0(f_1,f_2)\rangle.$$

Notation A.2.7. For a morphism $f : \mathbf{a} \to \mathbf{b}$ in $\mathbf{Alg}_\tau$ set

$$\mathbf{eq}(f) = \{(x, y) \in \mathbf{a} \times \mathbf{a} \mid f(x) = f(y)\}\,.$$

Proposition A.2.26. *For a pair $f_1 : \mathbf{a} \to \mathbf{b}$, $f_2 : \mathbf{a} \to \mathbf{b}$ of parallel morphisms in $\mathbf{Alg}_\tau$ let $\mathbf{e} = \mathbf{r}(f_1, f_2)$ and*

$$\begin{aligned} j : \mathbf{b} &\to \mathbf{b}/\mathbf{e} \\ x &\mapsto [x]_\mathbf{e}. \end{aligned}$$

Then j is a direct equalizer for the pair f_1, f_2.

Proof. A straightforward consequence of Proposition A.2.25. ✠

Corollary A.2.2. *In $\mathbf{Alg}_\tau$ there is a direct equalizer for any pair of parallel morphisms.*

Proposition A.2.27. *In $\mathbf{Alg}_\tau$ the morphism $j : \mathbf{b} \to \mathbf{e}$ is a direct equalizer for a pair $f_1 : \mathbf{a} \to \mathbf{b}$ and $f_2 : \mathbf{a} \to \mathbf{b}$ of parallel morphisms if and only if it is surjective and* $\mathrm{Coi}\, j = \mathbf{b}/\mathbf{r}(f_1, f_2)$.

Proof. Let j be a direct equalizer for the pair f_1, f_2. Since

$$\begin{aligned} i : \mathbf{b} &\to \mathbf{b}/\mathbf{r}(f_1, f_2) \\ x &\mapsto [x]_{\mathbf{r}(f_1,f_2)} \end{aligned}$$

is also a direct equalizer for f_1 and f_2, there is an isomorphism $h : \mathbf{e} \to \mathbf{b}/\mathbf{r}(f_1, f_2)$ and $j = h \circ i$. Thus j is surjective and $\mathrm{Coi}(j) = \mathrm{Coi}(i) = \mathbf{b}/\mathbf{r}(f_1, f_2)$.
Let j be surjective and $\mathrm{Coi}(j) = \mathbf{b}/\mathbf{r}(f_1, f_2)$. Then $j \circ f_1 = j \circ f_2$. If $h : \mathbf{b} \to \mathbf{f}$ is such that $h \circ f_1 = h \circ f_2$, then let $t \in \mathbf{e}$; if $s_1 \in \mathbf{b}$ and $s_2 \in \mathbf{b}$ are such that $j(s_1) = j(s_2)$, then $(s_1, s_2) \in \mathbf{eq}(j) = \mathbf{r}(f_1, f_2)$, thus, since $\mathbf{r}(f_1, f_2) \subseteq \mathbf{eq}(h)$, $h(s_1) = h(s_2)$. So we can define

$$\begin{aligned} h' : \mathbf{e} &\to \mathbf{f} \\ t &\mapsto h(s) \end{aligned}$$

where $s \in \mathbf{b}$ is such that $t = j(s)$. It follows that $h = h' \circ j$. If $k : \mathbf{e} \to \mathbf{f}$ is such that $h = k \circ j$, let $y \in \mathbf{e}$; then $y = j(s)$ for some $s \in \mathbf{e}$, thus $k(y) = k(j(s)) = h(s)$, so $k = h'$. ✠

Definition A.2.21. Let $\mathbf{C}$ be a category with a null object, $f \in \overline{\mathbf{C}}(\mathbf{a}, \mathbf{b})$. A ***cokernel*** of f is a direct equalizer for the pair of parallel morphisms f and $\mathbf{0}^\mathbf{a}_\mathbf{b}$. We shall denote with $\mathrm{cok}(f)$ any cokernel of f, and call $\mathrm{cod}(\mathrm{cok}(f))$ a ***cokernel object*** of f.

Proposition A.2.28. *If $\mathbf{C}$ is a category with a null object $\mathbf{0}$ and $f \in \mathcal{M}(\mathbf{C})$ is an epimorphism, then $\mathbf{0}^{\mathrm{cod}(f)}_\mathbf{0}$ is a cokernel of f. It follows that any cokernel object of f is a null object.*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$, and let $g : \mathbf{b} \to \mathbf{c}$ be such that $g \circ f = \mathbf{0}^{\mathbf{a}}_{\mathbf{c}}$, thus $g \circ f = \mathbf{0}^{\mathbf{b}}_{\mathbf{c}} \circ f$; since f is an epimorphism, this yields $g = \mathbf{0}^{\mathbf{b}}_{\mathbf{c}}$, thus there is a unique $\mathbf{0}^{\mathbf{0}}_{\mathbf{c}} : \mathbf{0} \to \mathbf{c}$ such that $g = \mathbf{0}^{\mathbf{0}}_{\mathbf{c}} \circ \mathbf{0}^{\mathbf{b}}_{\mathbf{0}}$. ✠

Proposition A.2.29. *If* $\mathbf{C}$ *is an ab-category with a null object and* $f \in \mathcal{M}(\mathbf{C})$ *is such that any cokernel object of* f *is a null object, then* f *is an epimorphism*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$, and $g : \mathbf{b} \to \mathbf{c}$, $h : \mathbf{b} \to \mathbf{c}$ such that $g \circ f = h \circ f$. Then $g \circ f - h \circ f = \mathbf{0}^{\mathbf{a}}_{\mathbf{c}}$, whence $(g - h) \circ f = \mathbf{0}^{\mathbf{a}}_{\mathbf{c}}$, $g - h = \mathbf{0}^{\mathbf{b}}_{\mathbf{c}}$ and $g = h$. ✠

Proposition A.2.30. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathrm{F}, \mathrm{G} : \mathbf{C} \to \mathbf{D}$, $\mathrm{H}, \mathrm{K} : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}^{\mathbf{C}}$, $\mathrm{f} : \mathrm{F} \to \mathrm{G}$, $\mathrm{h} : \mathrm{H} \to \mathrm{K}$. *If the functors* $\mathrm{H}(\mathrm{F})$, $\mathrm{H}(\mathrm{G})$, $\mathrm{K}(\mathrm{F})$, $\mathrm{K}(\mathrm{G})$ *have direct limits respectively* $l^{\mathrm{HF}} = (\mathbf{l}^{\mathrm{HF}}, \lambda^{\mathrm{HF}})$, $l^{\mathrm{HG}} = (\mathbf{l}^{\mathrm{HG}}, \lambda^{\mathrm{HG}})$, $l^{\mathrm{KF}} = (\mathbf{l}^{\mathrm{KF}}, \lambda^{\mathrm{KF}})$, $l^{\mathrm{KG}} = (\mathbf{l}^{\mathrm{KG}}, \lambda^{\mathrm{KG}})$, *the diagram*

$$\begin{array}{ccc} \mathbf{l}^{\mathrm{HF}} & \xrightarrow{\varinjlim[\mathrm{H}(\mathrm{f}), l^{\mathrm{HF}}, l^{\mathrm{HG}}]} & \mathbf{l}^{\mathrm{HG}} \\ {\scriptstyle \varinjlim[\mathrm{h}_{\mathrm{F}}, l^{\mathrm{HF}}, l^{\mathrm{KF}}]} \big\downarrow & & \big\downarrow {\scriptstyle \varinjlim[\mathrm{h}_{\mathrm{G}}, l^{\mathrm{HG}}, l^{\mathrm{KG}}]} \\ \mathbf{l}^{\mathrm{KF}} & \xrightarrow{\varinjlim[\mathrm{K}(\mathrm{f}), l^{\mathrm{KF}}, l^{\mathrm{KG}}]} & \mathbf{l}^{\mathrm{KG}} \end{array}$$

commutes.

Proof. For each $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ we have

$$\begin{aligned} \varinjlim[\mathrm{K}(\mathrm{f}), l^{\mathrm{KF}}, l^{\mathrm{KG}}] \circ \varinjlim[\mathrm{h}_{\mathrm{F}}, l^{\mathrm{HF}}, l^{\mathrm{KF}}] \circ \lambda^{\mathrm{HF}}_{\mathbf{c}} &= \varinjlim[\mathrm{K}(\mathrm{f}), l^{\mathrm{KF}}, l^{\mathrm{KG}}] \circ \lambda^{\mathrm{KF}}_{\mathbf{c}} \circ (\mathrm{h}_{\mathrm{F}})_{\mathbf{c}} = \\ &= \lambda^{\mathrm{KG}}_{\mathbf{c}} \circ \mathrm{K}(\mathrm{f})_{\mathbf{c}} \circ (\mathrm{h}_{\mathrm{F}})_{\mathbf{c}} = \\ &= \lambda^{\mathrm{KG}}_{\mathbf{c}} \circ (\mathrm{h}_{\mathrm{G}})_{\mathbf{c}} \circ \mathrm{H}(\mathrm{f})_{\mathbf{c}} = \\ &= \varinjlim[\mathrm{h}_{\mathrm{G}}, l^{\mathrm{HG}}, l^{\mathrm{KG}}] \circ \lambda^{\mathrm{HG}}_{\mathbf{c}} \circ \mathrm{H}(\mathrm{f})_{\mathbf{c}} = \\ &= \varinjlim[\mathrm{h}_{\mathrm{G}}, l^{\mathrm{HG}}, l^{\mathrm{KG}}] \circ \varinjlim[\mathrm{H}(\mathrm{f}), l^{\mathrm{HF}}, l^{\mathrm{HG}}] \circ \lambda^{\mathrm{HF}}_{\mathbf{c}} \end{aligned}$$

whence the thesis follows. ✠

Theorem A.2.1 (Construction of direct limits by direct products and binary direct equalisers). *For categories* $\mathbf{C}$, $\mathbf{D}$, *if* $\mathbf{D}$ *has binary direct equalisers and every functor from* $\mathcal{O}(\mathbf{C})^*$ *and from* $\mathcal{M}(\mathbf{C})^*$ *to* $\mathbf{D}$ *has a direct product, then every functor from* $\mathbf{C}$ *to* $\mathbf{D}$ *has a direct limit.*
In particular, let $\mathrm{F}^1 : \mathcal{O}(\mathbf{C})^* \to \mathbf{D}$ *be defined by*

$$\mathrm{F}^1(\mathbf{c}) = \mathrm{F}(\mathbf{c}) \tag{A.48}$$

and $\mathrm{F}^2 : \mathcal{M}(\mathbf{C})^* \to \mathbf{D}$ *by*

$$\mathrm{F}^2(u) = \mathrm{F}(\mathrm{dom}(u)); \tag{A.49}$$

let $f_1, f_2 : \coprod \mathrm{F}^2 \to \coprod \mathrm{F}^1$ *be defined by*

$$i^{\coprod \mathrm{F}^1}_{\mathrm{dom}(u)} = f_1 i^{\coprod \mathrm{F}^2}_{u} \text{ for } u \in \mathcal{M}(\mathbf{C}) \tag{A.50}$$

and

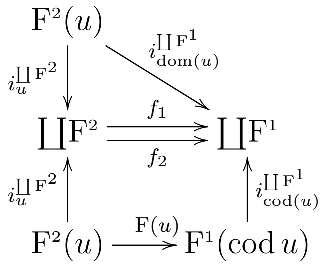
$$i^{\coprod \mathrm{F}^1}_{\mathrm{cod}(u)}\mathrm{F}(u) = f_2 i^{\coprod \mathrm{F}^2}_{u} \text{ for } u \in \mathcal{M}(\mathbf{C}) \tag{A.51}$$

and let $(\mathbf{e}, k)$ *be a direct equalizer for* f_1 *and* f_2*, and*

$$m_{\mathbf{c}} = k\, i^{\coprod \mathrm{F}^1}_{\mathbf{c}} \text{ for } \mathbf{c} \in \mathcal{O}(\mathbf{C}). \tag{A.52}$$

Then there is a natural transformation $\epsilon : \mathrm{F} \to \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{e}}$ *such that for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\epsilon_{\mathbf{c}} = m_{\mathbf{c}}$ *and* $(\mathbf{e}, \epsilon)$ *is a direct limit of* F*.*

Proof. As shown in the commutative diagram



(A.53)

for $u \in \mathcal{M}(\mathbf{C})$ both $i^{\coprod \mathrm{F}^1}_{\mathrm{dom}\, u}$ and $\mathrm{F}(u) i^{\coprod \mathrm{F}^1}_{\mathrm{cod}\, u}$ factor uniquely through $i^{\coprod \mathrm{F}^2}_{u}$, so there are f_1, f_2 such that

$$i^{\coprod \mathrm{F}^1}_{\mathrm{dom}\, u} = f_1 i^{\coprod \mathrm{F}^1}_{u} \tag{A.54}$$

and

$$i^{\coprod \mathrm{F}^1}_{\mathrm{cod}\, u}\mathrm{F}(u) = f_2 i^{\coprod \mathrm{F}^2}_{u} \tag{A.55}$$

for each $u \in \mathcal{M}(\mathbf{C})$.
If $v \in \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$

$$m_{\mathbf{c}_2}\mathrm{F}(v) = k\, i^{\coprod \mathrm{F}^1}_{\mathbf{c}_2}\mathrm{F}(v) = k\, f_2\, i^{\coprod \mathrm{F}^2}_{v} = k\, f_1\, i^{\coprod \mathrm{F}^1}_{v} = k\, i^{\coprod \mathrm{F}^1}_{\mathbf{c}_1} = m_{\mathbf{c}_1} \tag{A.56}$$

so there is a natural transformation $\epsilon : \mathrm{F} \to \mathrm{C}^{\mathbf{C},\mathbf{D}}_{\mathbf{e}}$ such that for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\epsilon_{\mathbf{c}} = m_{\mathbf{c}}$, and $(\mathbf{e}, \epsilon)$ is a direct target of F.
If $(\mathbf{t}, \tau)$ is another direct target for F then τ as a natural transformation in $\mathbf{D}^{\mathcal{O}(\mathbf{C})^*}$ factors through $\gamma_h^{\mathcal{O}(\mathbf{C})^*,\mathbf{D}}$ for a unique morphism $h : \coprod \mathrm{F}^1 \to \mathbf{t}$, as F coincides with F^1 on the objects. But τ is also a natural transformation in $\mathbf{D}^{\mathbf{C}}$, so for each $v : \mathbf{c}_1 \to \mathbf{c}_2$ we have $\tau_{\mathbf{c}_1} = \tau_{\mathbf{c}_2}\mathrm{F}(v)$ and thus

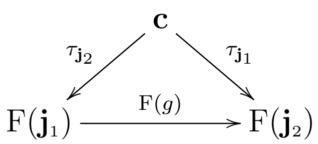
$$\tau_{\mathbf{c}_1} = h i^{\coprod \mathrm{F}^1}_{\mathbf{c}_1} = h f_1 i^{\coprod \mathrm{F}^2}_{v} = \tau_{\mathbf{c}_2}\mathrm{F}(v) = h i^{\coprod \mathrm{F}^1}_{\mathbf{c}_2}\mathrm{F}(v) = h f_2 i^{\coprod \mathrm{F}^2}_{v} \tag{A.57}$$

whence $hf_1 = hf_2$. Thus h factors uniquely through k and τ factors uniquely through ϵ. ✠

A.2.2 Inverse limits

Definition A.2.22. Let $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ be a functor. An ***inverse target for*** F is an arrow from the diagonal functor $\Delta^{\mathcal{C}}_{\mathcal{J}}$ to F.

Remark A.2.5. According to this definition, an inverse target for the functor $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ is a pair $(\mathbf{c},\tau)$ where $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ and τ is a natural transformation $\tau:\mathrm{C}_{\mathbf{c}}\to\mathrm{F}$. That is, for $\mathbf{j}\in\mathcal{O}(\mathbf{J})$, $\tau_{\mathbf{j}}:\mathbf{c}\to F(\mathbf{j})$, and for any $\mathbf{j}_1\in\mathcal{O}(\mathbf{J})$ and $\mathbf{j}_2\in\mathcal{O}(\mathbf{J})$ and any $g:\mathbf{j}_1\to\mathbf{j}_2$ the diagram



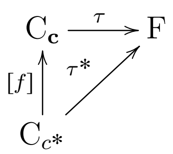
(A.58)

commutes.

Definition A.2.23. Let $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ be a functor. An ***inverse limit for*** F is a universal arrow from the diagonal functor $\Delta^{\mathcal{C}}_{\mathcal{J}}$ to F.

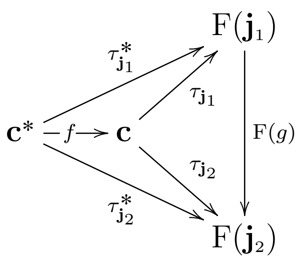
Definition A.2.24. If $(\mathbf{c},\tau)$ is an inverse limit for the functor F, then $\mathbf{c}$ is an ***inverse limit object*** for F and τ is an ***inverse limit cone*** for F.

Remark A.2.6. According to this definition, am inverse limit for the functor $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ is an inverse target $(\mathbf{c},\tau)$ for F such that if $(\mathbf{c}^*,\tau^*)$ is any inverse target for F, there is a unique morphism $f:\mathbf{c}^*\to\mathbf{c}$ such that $\tau^*=\tau\circ[f]$, that is the diagram



(A.59)

commutes. More in details, for any $\mathbf{j}_1\in\mathcal{O}(\mathbf{J})$ and $\mathbf{j}_2\in\mathcal{O}(\mathbf{J})$ and any $g:\mathbf{j}_1\to\mathbf{j}_2$ the diagram



(A.60)

commutes.

Proposition A.2.31. *Any two inverse limits for a functor are isomorphic.*

Proof. A direct consequence of Proposition A.2.15 ✠

Notation A.2.8. We will write

$$\varprojlim \mathrm{F} \tag{A.61}$$

for the class of isomorphism of the inverse limit of the functor F.

Proposition A.2.32. *Let* $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ *be a functor. If* $\mathbf{J}$ *has an initial object* $\mathbf{i}$*, for* $\mathbf{j}\in\mathcal{O}(\mathbf{J})$ *let* $f_{\mathbf{j}}$ *be the unique element of* $\overline{\mathbf{J}}(\mathbf{i},\mathbf{j})$*,* $\mathbf{i}^* = F(\mathbf{i})$*, and* $\tau : \mathrm{C}_{\mathbf{i}^*} \to \mathrm{F}$ *the natural transformation defined, for* $\mathbf{j}\in\mathcal{O}(\mathbf{J})$*, by* $\tau_{\mathbf{j}} = \mathrm{F}(f_{\mathbf{j}})$*. Then* $(\mathbf{i}^*,\tau)$ *is an inverse limit for* F*.*

Proof. Of course $(\mathbf{i}^*,\tau)$ is an inverse target for F. Let $(\mathbf{s},\sigma_{\mathbf{j}})$ be an inverse target for F. Then $\sigma_{\mathbf{i}} : \mathbf{s} \to \mathbf{i}^*$ and for $\mathbf{j}\in\mathcal{O}(\mathbf{J})$ we have $\sigma_{\mathbf{j}} = \tau_{\mathbf{j}}\circ\sigma_{\mathbf{i}}$. If $g : \mathbf{s}\to\mathbf{i}^*$ is such that for $\mathbf{j}\in\mathcal{O}(\mathbf{J})$ we have $\sigma_{\mathbf{j}} = \tau_{\mathbf{j}}\circ g$, then in particular $\sigma_{\mathbf{i}} = \tau_{\mathbf{i}}\circ g$, but $\tau_{\mathbf{i}} = \mathrm{F}(f_{\mathbf{i}}) = \mathrm{F}(\mathrm{id}_{\mathbf{i}}) = \mathrm{id}_{\mathbf{i}^*}$, so $g = \sigma_{\mathbf{i}}$. ✠

Definition A.2.25. Let $\mathrm{F}:\mathbf{A}\to\mathbf{B}$, $\mathrm{F}:\mathbf{B}\to\mathbf{C}$ be functors. We say that G ***creates inverse limits for*** F if for every inverse limit $(\mathbf{c},\tau)$ of $\mathrm{G}\circ\mathrm{F}$ there exists a unique inverse limit $(\mathbf{b},\sigma)$ of F such that

- $G(\mathbf{b}) = \mathbf{c}$
- $\forall\mathbf{a}\in\mathcal{O}(\mathbf{A})\ \tau_{\mathbf{a}} = G(\sigma_{\mathbf{a}})$.

Definition A.2.26. An ***inverse product*** in a category $\mathbf{C}$ is an inverse limit for a functor $\mathrm{F}:\mathbf{I}\to\mathbf{C}$ where $\mathbf{I}$ is a discrete category.

Notation A.2.9. The isomorphism class of inverse products for the functor F will be noted by

$$\prod \mathrm{F}.$$

Remark A.2.7. Being a discrete category essentially a set, a functor having a discrete category as its domain can be described as a ***collection of objects*** in the codomain category. Thus $(\mathbf{c}_i)_{i\in\mathrm{I}}$ means that the $\mathbf{c}_i$ are objects of a category, say $\mathbf{C}$, $\mathbf{I}$ is a set, and there is a functor, say $\mathrm{F}:\mathbf{I}^*\to\mathbf{C}$, such that $\mathbf{c}_i = \mathrm{F}(i)$ for each $i\in\mathbf{I}$.
A natural transformation between two collections of objects $(\mathbf{c}_i)_{i\in\mathrm{I}}$ and $(\mathbf{d}_i)_{i\in\mathrm{I}}$ of the same category is just a collection of morfisms $(f_i)_{i\in\mathrm{I}}$ such that $f_i : \mathbf{c}_i\to\mathbf{d}_i$ for each $i\in\mathrm{I}$.
An inverse product of $(\mathbf{c}_i)_{i\in\mathrm{I}}$ is a pair $(\mathbf{p},\tau)$, where $\mathbf{p}\in\mathcal{O}(\mathbf{C})$ and $\tau = (\tau_i)_{i\in\mathrm{I}}$ is a collection of morphisms $\tau_i : \mathbf{p}\to\mathbf{c}_i$, such that for any pair $(\mathbf{q},\sigma)$ where $\mathbf{q}\in\mathcal{O}(\mathbf{C})$ and $\sigma = (\sigma_i)_{i\in\mathrm{I}}$ with $\sigma_i : \mathbf{q}\to\mathbf{c}_i$, there is a unique morphism $f : \mathbf{q}\to\mathbf{p}$ such that $\sigma_i = \tau_i\circ f$ for each $i\in\mathrm{I}$.
Using this notation the isomorphism class of inverse products for the collection $(\mathbf{c}_i)_{i\in I}$ will be noted by

$$\prod_{i\in\mathrm{I}}\mathbf{c}_i.$$

If

$$(\mathbf{p}, \pi) \in \prod_{i \in \mathrm{I}} \mathbf{c}_i,$$

$$(\mathbf{r}, \rho) \in \prod_{i \in \mathrm{I}} \mathbf{d}_i,$$

and τ is a natural transformation between $(\mathbf{c}_i)_{i\in\mathrm{I}}$ and $(\mathbf{d}_i)_{i\in\mathrm{I}}$, then its inverse limit relative to $(\mathbf{p}, \pi)$ and $(\mathbf{r}, \rho)$ is a morphism $t : \mathbf{p} \to \mathbf{r}$ such that $\tau_i \circ \pi_i = \rho_i \circ \tau$ for each $i \in \mathrm{I}$.

Remark A.2.8. In **Set**, a collection $(A_i)_{i\in\mathrm{I}}$ has always an inverse product $(\prod_{i\in\mathrm{I}} A_i, p)$ where $\prod_{i\in\mathrm{I}} A_i$ is the usual cartesian product of the sets A_i and $p = (p_i)_{i\in\mathrm{I}}$ is the collection of the projections onto the factors A_i. That is, $\prod_{i\in\mathrm{I}} A_i$ is the set of maps

$$f : \mathrm{I} \to \bigcup_{i\in\mathrm{I}} A_i$$

$$i \mapsto f_i$$

such that $f_i \in A_i$ for $i \in \mathrm{I}$; such an element of $\prod_{i\in\mathrm{I}} A_i$ is usually written as $(f_i)_{i\in\mathrm{I}}$; and

$$p_j : \prod_{i\in\mathrm{I}} A_i \to A_j$$

$$(f_i)_{i\in\mathrm{I}} \mapsto f_j.$$

It is easily verified that $(\prod_{i\in\mathrm{I}} A_i, (p_i)_{i\in\mathrm{I}})$ has the required properties for an inverse product of $(A_i)_{i\in\mathrm{I}}$ in **Set**.

Proposition A.2.33. *Let I be a set,* **C** *a category with inverse products, and* $\mathrm{F}:I^* \times I^* \to \mathbf{C}$. *For $i \in I$ set* $\mathrm{F}_i : I^* \to \mathbf{C}$ *by* $\mathrm{F}_i(j) = \mathrm{F}(i,j)$ *for* $(j \in I)$, *and* $(\pi_i, \sigma_i) \in \prod \mathrm{F}_i$. *Set* $\mathrm{F}' : I^* \to \mathbf{C}$ *as* $\mathrm{F}'(i) = \pi_i$ *for* $i \in I$ *and let* $(\alpha, \tau) \in \prod \mathrm{F}'$. *Set* $\rho : \mathrm{C}_{\alpha}^{I^* \times I^*, \mathbf{C}} \to \mathrm{F}$ *defined for* $(i,j) \in I \times I$ *by* $\rho_{(i,j)} = (\sigma_i)_j \tau_i$.
Then $(\alpha, \rho) \in \prod \mathrm{F}$.

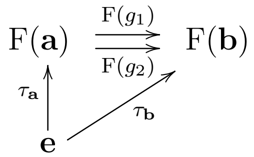
Proof. Let $(\beta, \mu) \in \prod \mathrm{F}$. It will suffice to show that μ factors through ρ.
For $i \in I$ set $\varepsilon_i : \mathrm{C}_{\beta}^{I^*, \mathbf{C}} \to \mathrm{F}_i$ defined for $j \in I$ by $(\varepsilon_i)_j = \mu_{(i,j)}$. Then each ε_i factors uniquely through σ_i, $\varepsilon_i = \sigma_i \varphi_i$. Since each $\varphi_i : \mathrm{C}_{\beta}^{I^*, \mathbf{C}} \to \mathrm{C}_{\pi_i}^{I^*, \mathbf{C}}$ is a natural transformation between constant functors, it is a constant natural transformation, that is there is a morphism $\varphi_i' : \beta \to \pi_i$ such that $(\varphi_i)_j = \varphi_i'$ for $j \in I$, and these morphisms define a natural transformation $\varphi' : \mathrm{C}_{\beta}^{I^*, \mathbf{C}} \to \mathrm{F}'$ which factors uniquely through τ, $h' = \tau k$, where $k : \mathrm{C}_{\beta}^{I^*, \mathbf{C}} \to \mathrm{C}_{\alpha}^{I^*, \mathbf{C}}$ is also a constant natural transformation, so there is $k' : \beta \to \alpha$ such that $k_i = k'$ for $i \in I$. Thus $h_i' = \tau_i k'$ for each $i \in I$, and $(h_i')_j = \tau_i k'$ for each $i, j \in I$, thus $(\varepsilon_i)_j = (\sigma_i)_j \tau_i k'$, that is, $\mu = \rho \gamma_{k'}^{I^* \times I^*, \mathbf{C}}$. ✠

Definition A.2.27. An ***inverse equalizer*** in a category **C** is an inverse limit for a functor $\mathrm{F}:\mathbf{J} \to \mathbf{C}$ where **J** is a category of type $\{\cdot \rightrightarrows \cdot\}$.

Remark A.2.9. Let $\mathrm{F}:\mathbf{J}\to\mathbf{C}$ where

1. $\mathcal{O}(\mathbf{J}) = \{\mathbf{a}, \mathbf{b}\}$
2. $\overline{\mathbf{J}}(\mathbf{a}, \mathbf{b}) = \{g_1, g_2\}$

and let $(\mathbf{e}, \tau)$ be an inverse equalizer for F. Then the following diagram commutes



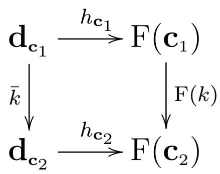
that is, $\tau_{\mathbf{b}} = \mathrm{F}(g_1) \circ \tau_{\mathbf{a}} = \mathrm{F}(g_2) \circ \tau_{\mathbf{a}}$, and if $(\mathbf{f}, \sigma)$ is an inverse target for F, which means that $\sigma_{\mathbf{b}} = \mathrm{F}(g_1) \circ \sigma_{\mathbf{a}} = \mathrm{F}(g_2) \circ \sigma_{\mathbf{a}}$, then there is a unique morphism $h : \mathbf{f} \to \mathbf{e}$ such that $\sigma_{\mathbf{a}} = \tau_{\mathbf{a}} \circ h$ and $\sigma_{\mathbf{b}} = \tau_{\mathbf{b}} \circ h$. So we can restate the definition of inverse equalizer in the following terms, that refer to a pair of parallel morphisms rather than to a functor. The morphism $i : \mathbf{e} \to \mathbf{c}$ is an inverse equalizer for the pair of parallel morphisms $f_1 : \mathbf{c} \to \mathbf{d}$ and $f_2 : \mathbf{c} \to \mathbf{d}$ if

- $f_1 \circ i = f_2 \circ i$
- for any morphism $j : \mathbf{f} \to \mathbf{c}$ such that $f_1 \circ j = f_2 \circ j$ there is a unique morphism $h : \mathbf{f} \to \mathbf{e}$ such that $j = i \circ h$.

Proposition A.2.34. *If $i : \mathbf{e} \to \mathbf{a}$ is an inverse equalizer then i is monic.*

Proof. Let $i : \mathbf{e} \to \mathbf{a}$ be an inverse equalizer for the pair $f_1 : \mathbf{a} \to \mathbf{b}$ and $f_2 : \mathbf{a} \to \mathbf{b}$. Given a morphism $g : \mathbf{f} \to \mathbf{e}$, for $h = i \circ g$ also $f_1 \circ h = f_2 \circ h$ holds, so g is the only morphism such that $h = i \circ g$. Then $i \circ g_1 = i \circ g_2$ implies $g_1 = g_2$. ✠

Lemma A.2.7. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$, $\mathrm{G}:\mathbf{C}\to\mathbf{D}$, $\mathrm{f} : \mathrm{F} \to \mathrm{G}$, $\mathrm{g} : \mathrm{F} \to \mathrm{G}$ *and for each* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *let* $h_{\mathbf{c}} : \mathbf{d}_{\mathbf{c}} \to \mathrm{F}(\mathbf{c})$ *be an inverse equalizer for the pair* $\mathrm{f}_{\mathbf{c}}, \mathrm{g}_{\mathbf{c}}$*. Then for any morphism* $k : \mathbf{c}_1 \to \mathbf{c}_2$ *there is a unique morphism* $\bar{k} : \mathbf{d}_{\mathbf{c}_1} \to \mathbf{d}_{\mathbf{c}_2}$ *such that the diagram*



commutes, that is $h_{\mathbf{c}_2} \circ \bar{k} = \mathrm{F}(k) \circ h_{\mathbf{c}_1}$*. In particular for each* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$, $\overline{\mathrm{id}_{\mathbf{c}}} = \mathrm{id}_{\mathbf{d}_{\mathbf{c}}}$, *and for each pair of composable morphisms* $k_1 : \mathbf{c}_1 \to \mathbf{c}_2$, $k_2 : \mathbf{c}_2 \to \mathbf{c}_3$, $\overline{k_2 \circ k_1} = \bar{k}_2 \circ \bar{k}_1$.

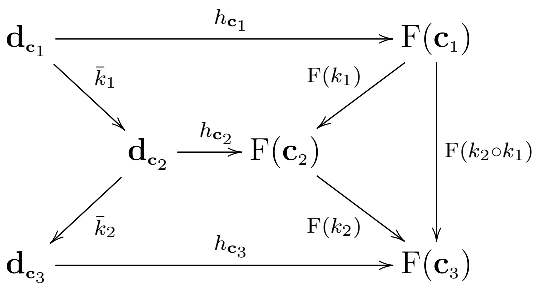
Proof. We have

$$\mathrm{f}_{\mathbf{c}_2} \circ \mathrm{F}(k) \circ h_{\mathbf{c}_1} = \mathrm{G}(k) \circ \mathrm{f}_{\mathbf{c}_1} \circ h_{\mathbf{c}_1} = \mathrm{G}(k) \circ \mathrm{g}_{\mathbf{c}_1} \circ h_{\mathbf{c}_1} = \mathrm{g}_{\mathbf{c}_2} \circ \mathrm{F}(k) \circ h_{\mathbf{c}_1}$$

so there is a unique morphism $\bar{k} : \mathbf{d}_{\mathbf{c}_1} \to \mathbf{d}_{\mathbf{c}_2}$ such that $\mathrm{F}(k) \circ h_{\mathbf{c}_1} = h_{\mathbf{c}_2} \circ \bar{k}$.
Since the diagram

$$\begin{array}{ccc} \mathbf{d}_{\mathbf{c}} & \xrightarrow{h_{\mathbf{c}}} & \mathrm{F}(\mathbf{c}) \\ {\scriptstyle \mathrm{id}_{\mathbf{d}_{\mathbf{c}}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{F}(\mathbf{c})}} \\ \mathbf{d}_{\mathbf{c}} & \xrightarrow{h_{\mathbf{c}}} & \mathrm{F}(\mathbf{c}) \end{array}$$

commutes, then $\overline{\mathrm{id}}_{\mathbf{c}} = \mathrm{id}_{\mathbf{d}_{\mathbf{c}}}$.
The diagram



commutes, so

$$\mathrm{F}(k_2 \circ k_1) \circ h_{\mathbf{c}_1} = \mathrm{F}(k_2) \circ \mathrm{F}(k_1) \circ h_{\mathbf{c}_1} = \mathrm{F}(k_2) \circ h_{\mathbf{c}_2} \circ \bar{k}_1 = h_{\mathbf{c}_3} \circ \bar{k}_2 \circ \bar{k}_1$$

that is, $\overline{k_2 \circ k_1} = \bar{k}_2 \circ \bar{k}_1$. ✠

Proposition A.2.35. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $\mathrm{F}:\mathbf{C} \to \mathbf{D}$*,* $\mathrm{G}:\mathbf{C} \to \mathbf{D}$*,* $\mathrm{f} : \mathrm{F} \to \mathrm{G}$*,* $\mathrm{g} : \mathrm{F} \to \mathrm{G}$ *and for each* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *let* $h_{\mathbf{c}} : \mathbf{d}_{\mathbf{c}} \to \mathrm{F}(\mathbf{c})$ *be an inverse equalizer for the pair* $\mathrm{f}_{\mathbf{c}}, \mathrm{g}_{\mathbf{c}}$*. Then there is a natural transformation* $\mathrm{h} : \mathrm{H} \to \mathrm{F}$ *that is an inverse equalizer for the pair* f, g *and such that* $\mathrm{h}_{\mathbf{c}} = h_{\mathbf{c}}$*.*

Proof. That the functor H exists, and that h is a natural transformation, is stated by Lemma A.2.7. Let's prove that h is an inverse equalizer for f, g. If $\mathrm{k} : \mathrm{K} \to \mathrm{F}$ is such that $\mathrm{f} \circ \mathrm{k} = \mathrm{g} \circ \mathrm{k}$, then, for each $\mathbf{c} \in \mathcal{O}(\mathbf{C})$, $\mathrm{f}_{\mathbf{c}} \circ \mathrm{k}_{\mathbf{c}} = \mathrm{g}_{\mathbf{c}} \circ \mathrm{k}_{\mathbf{c}}$, so there is a unique $i_{\mathbf{c}} : \mathrm{K}(\mathbf{c}) \to \mathrm{H}(\mathbf{c})$ such that $\mathrm{k}_{\mathbf{c}} = \mathrm{h}_{\mathbf{c}} \circ i_{\mathbf{c}}$. The morphisms $i_{\mathbf{c}}$ are the component of a natural transformation from K to H since, for a morphism $j \in \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$, the diagram

$$\begin{array}{ccccc} \mathrm{K}(\mathbf{c}_1) & \xrightarrow{i_{\mathbf{c}_1}} & \mathrm{H}(\mathbf{c}_1) & \xrightarrow{\mathrm{h}_{\mathbf{c}_1}} & \mathrm{F}(\mathbf{c}_1) \\ {\scriptstyle \mathrm{K}(j)}\downarrow & & {\scriptstyle \mathrm{H}(j)}\downarrow & & \downarrow{\scriptstyle \mathrm{F}(j)} \\ \mathrm{K}(\mathbf{c}_2) & \xrightarrow{i_{\mathbf{c}_2}} & \mathrm{H}(\mathbf{c}_2) & \xrightarrow{\mathrm{h}_{\mathbf{c}_2}} & \mathrm{F}(\mathbf{c}_2) \end{array}$$

commutes because

$$\mathrm{h}_{\mathbf{c}_2} \circ \mathrm{H}(j) \circ i_{\mathbf{c}_1} = \mathrm{F}(j) \circ \mathrm{h}_{\mathbf{c}_1} \circ i_{\mathbf{c}_1} = \mathrm{F}(j) \circ \mathrm{k}_{\mathbf{c}_1} = \mathrm{k}_{\mathbf{c}_2} \circ \mathrm{K}(j) = \mathrm{h}_{\mathbf{c}_2} \circ i_{\mathbf{c}_2} \circ \mathrm{K}(j)$$

whence, since $\mathrm{h}_{\mathbf{c}_2}$ is a monomorphism, $\mathrm{H}(j) \circ i_{\mathbf{c}_1} = i_{\mathbf{c}_2} \circ \mathrm{K}(j)$. ✠

Notation A.2.10. For a pair $f_1 : \mathbf{a} \to \mathbf{b}$ and $f_2 : \mathbf{a} \to \mathbf{b}$ of parallel morphisms in $\mathbf{Alg}_\tau$ let $\mathbf{s}(f_1, f_2)$ be the subalgebra of $\mathbf{a}$ defined by

$$\mathbf{s}(f_1, f_2) = \{x \in \mathbf{a} \mid f_1(x) = f_2(x)\} . \tag{A.62}$$

Proposition A.2.36. *For a pair $f_1 : \mathbf{a} \to \mathbf{b}$ and $f_2 : \mathbf{a} \to \mathbf{b}$ of parallel morphisms in $\mathbf{Alg}_\tau$ let $\mathbf{e} = \mathbf{s}(f_1, f_2)$ and*

$$\begin{aligned} i : \mathbf{e} &\to \mathbf{a} \\ x &\mapsto x. \end{aligned} \tag{A.63}$$

Then i is an inverse equalizer for f_1 and f_2.

Proof. Of course $f_1 \circ i = f_2 \circ i$. If $j : \mathbf{d} \to \mathbf{a}$ is such that $f_1 \circ j = f_2 \circ j$ then $\operatorname{Img} j \subseteq \mathbf{e}$. Let

$$\begin{aligned} h : \mathbf{d} &\to \mathbf{e} \\ t &\mapsto j(t). \end{aligned}$$

Then $j = i \circ h$. If $j = i \circ g$, for $t \in \mathbf{d}$ we have $j(t) = i(g(t)) = g(t)$ so $g = h$. ✠

Corollary A.2.3. *In $\mathbf{Alg}_\tau$ there is an inverse equalizer for any pair of parallel morphisms.*

Proposition A.2.37. *In $\mathbf{Alg}_\tau$ the morphism $j : \mathbf{d} \to \mathbf{a}$ is an inverse equalizer for a pair $f_1 : \mathbf{a} \to \mathbf{b}$ and $f_2 : \mathbf{a} \to \mathbf{b}$ of parallel morphisms if and only if it is injective and* $\operatorname{Img} j = \mathbf{s}(f_1, f_2)$.

Proof. Let j be am inverse equalizer for the pair f_1, f_2. Since

$$\begin{aligned} i : \mathbf{s}(f_1, f_2) &\to \mathbf{a} \\ x &\mapsto x \end{aligned}$$

is also an inverse equalizer for f_1 and f_2, there is an isomorphism $h : \mathbf{d} \to (f_1, f_2)$ and $j = h \circ i$. Thus j is injective and $\operatorname{Img}(j) = \operatorname{Img}(i) = \mathbf{s}(f_1, f_2)$.
Let j be injective and $\operatorname{Img} j = \mathbf{s}(f_1, f_2)$. The latter implies $f_1 \circ j = f_2 \circ j$. If $k : \mathbf{f} \to \mathbf{a}$ is such that $f_1 \circ k = f_2 \circ k$, let $x \in \mathbf{f}$ and $u = k(x)$, so $f_1(u) = f_2(u)$, that is $u \in \operatorname{Img}(j)$, so there is a $y \in \mathbf{d}$ such that $u = j(y)$. This defines $g : \mathbf{f} \to \mathbf{d}$ such that $k = j \circ g$. If also $k = j \circ h$ then $g = h$, since j is injective. ✠

Definition A.2.28. Let $\mathbf{C}$ be a category with a null object, $f \in \overline{\mathbf{C}}(\mathbf{a}, \mathbf{b})$. A ***kernel*** of f is an inverse equalizer for the pair of parallel morphisms f and $\mathbf{0}_{\mathbf{b}}^{\mathbf{a}}$. We shall denote with $\ker(f)$ any kernel of f, and call $\operatorname{dom}(\ker(f))$ a ***kernel object*** of f.

Proposition A.2.38. *If $\mathbf{C}$ is a category with a null object and $f \in \mathcal{M}(\mathbf{C})$ is a monomorphism, then any kernel object of f is a null object.*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$, and let $g : \mathbf{c} \to \mathbf{a}$ be such that $f \circ g = \mathbf{0}^{\mathbf{c}}_{\mathbf{b}}$, thus $f \circ g = f \circ \mathbf{0}^{\mathbf{c}}_{\mathbf{a}}$; since f is a monomorphism, this yields $g = \mathbf{0}^{\mathbf{c}}_{\mathbf{a}}$, thus there is a unique $\mathbf{0}^{\mathbf{c}}_{\mathbf{0}} : \mathbf{c} \to \mathbf{0}$ such that $g = \mathbf{0}^{\mathbf{0}}_{\mathbf{a}} \circ \mathbf{0}^{\mathbf{c}}_{\mathbf{0}}$. ✠

Proposition A.2.39. *If* $\mathbf{C}$ *is an ab-category with a null object and* $f \in \mathcal{M}(\mathbf{C})$ *is such that any kernel object of* f *is a null object, then* f *is a monomorphism*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$, and $g : \mathbf{c} \to \mathbf{a}$, $h : \mathbf{c} \to \mathbf{a}$ such that $f \circ g = f \circ h$. Then $f \circ g - f \circ h = \mathbf{0}^{\mathbf{c}}_{\mathbf{b}}$, whence $f \circ (g - h) = \mathbf{0}^{\mathbf{c}}_{\mathbf{b}}$, $g - h = \mathbf{0}^{\mathbf{c}}_{\mathbf{a}}$ and $g = h$. ✠

Theorem A.2.2 (Pointwise construction of inverse limits in functor categories). *If every functor from* $\mathbf{J}$ *to* $\mathbf{D}$ *has an inverse limit then for every* $\mathbf{C}$ *every functor from* $\mathbf{J}$ *to* $\mathbf{D}^{\mathbf{C}}$ *has an inverse limit. Precisely, let* $\phi : \mathbf{J} \to \mathbf{D}^{\mathbf{C}}$. *For* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *define the functor* $\psi_{\mathbf{c}} : \mathbf{J} \to \mathbf{D}$ *by*

$$\psi_{\mathbf{c}}(\mathbf{j}) = \phi(\mathbf{j})(\mathbf{c}) \text{ for } \mathbf{j} \in \mathcal{O}(\mathbf{J}) \tag{A.64}$$

$$\psi_{\mathbf{c}}(f) = \phi(f)_{\mathbf{c}} \text{ for } f \in \mathcal{M}(\mathbf{J}) \tag{A.65}$$

and suppose $(\mathbf{l}_{\mathbf{c}}, \pi_{\mathbf{C}})$ *is an inverse limit for* $\psi_{\mathbf{c}}$. *Then* (Λ, γ) *is an inverse limit for* ϕ, *where the functor* $\Lambda : \mathbf{C} \to \mathbf{D}$ *is defined by*

$$\Lambda(\mathbf{c}) = \mathbf{l}_{\mathbf{c}} \tag{A.66}$$

and for $f : \mathbf{c}_1 \to \mathbf{c}_2$ $\Lambda(f)$ *is the limit of the natural transformation* $\Omega(f) : \psi_{\mathbf{c}_1} \to \psi_{\mathbf{c}_2}$ *defined for* $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ *by*

$$\Omega(f)_{\mathbf{j}} = \phi(\mathbf{j})(f), \tag{A.67}$$

and the natural transformation $\gamma : \mathrm{C}^{\mathbf{J}, \mathbf{D}^{\mathbf{C}}}_{\Lambda} \to \phi$ *is such that for* $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\gamma_{\mathbf{j}} : \Lambda \to \phi(\mathbf{j})$ *is the natural transformation defined for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *by*

$$(\gamma_{\mathbf{i}})_{\mathbf{c}} = (\pi_{\mathbf{c}})_{\mathbf{j}}. \tag{A.68}$$

Conversely, if (Λ, γ) *is an inverse limit of* $\phi : \mathbf{J} \to \mathbf{D}^{\mathbf{C}}$ *then for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $(\Lambda(\mathbf{c}), \pi_{\mathbf{c}})$, *where* $\pi_{\mathbf{c}}$ *are defined by A.68, is a limit of* $\psi_{\mathbf{c}}$ *as defined by A.64 and A.65.*

Proof. Let us show that for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\psi_{\mathbf{c}}$ is indeed a functor.
If $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ we have

$$\psi_{\mathbf{c}}(\mathrm{id}_{\mathbf{j}}) = \phi(\mathrm{id}_{\mathbf{j}})_{\mathbf{c}} = (\mathrm{id}_{\phi(\mathbf{j})})_{\mathbf{c}} = \mathrm{id}_{\phi(\mathbf{j})(\mathbf{c})} = \mathrm{id}_{\psi_{\mathbf{c}}(\mathbf{j})} \tag{A.69}$$

and for composable arrows g and f in $\mathcal{M}(\mathbf{J})$

$$\psi_{\mathbf{c}}(g \circ f) = \phi(g \circ g)_{\mathbf{c}} = \phi(g)_{\mathbf{c}} \circ \phi(f)_{\mathbf{c}} = \psi_{\mathbf{c}}(f) \circ \psi_{\mathbf{c}}(g). \tag{A.70}$$

Now to prove that Λ is a functor from $\mathbf{C}$ to $\mathbf{D}$ we need to prove first that for $f : \mathbf{c}_1 \to \mathbf{c}_2$ $\Omega(f) : \psi_{\mathbf{c}_1} \to \psi_{\mathbf{c}_2}$ is indeed a natural transformation.
For $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{O}(\mathbf{J})$ let $h : \mathbf{j}_1 \to \mathbf{j}_2$. We have to prove that the diagram

$$\begin{array}{ccc} \psi_{\mathbf{c}_1}(\mathbf{j}_1) & \xrightarrow{\Omega(f)_{\mathbf{j}_1}} & \psi_{\mathbf{c}_2}(\mathbf{j}_1) \\ {\scriptstyle \psi_{\mathbf{c}_1}(h)}\downarrow & & \downarrow{\scriptstyle \psi_{\mathbf{c}_2}(h)} \\ \psi_{\mathbf{c}_2}(\mathbf{j}_2) & \xrightarrow{\Omega(f)_{\mathbf{j}_2}} & \psi_{\mathbf{c}_2}(\mathbf{j}_2) \end{array} \tag{A.71}$$

commutes, which it does because it can be rewritten as

$$\begin{array}{ccc} \phi(\mathbf{j}_1)(\mathbf{c}_1) & \xrightarrow{\phi(h)_{\mathbf{c}_1}} & \phi(\mathbf{j}_2)(\mathbf{c}_1) \\ {\scriptstyle \phi(\mathbf{j}_1)(f)}\downarrow & & \downarrow{\scriptstyle \phi(\mathbf{j}_2)(f)} \\ \phi(\mathbf{j}_1)(\mathbf{c}_2) & \xrightarrow{\phi(h)_{\mathbf{c}_2}} & \phi(\mathbf{j}_2)(\mathbf{c}_2) \end{array} \tag{A.72}$$

and this commutes because $\phi(h)$ is by definition a natural transformation from $\phi(\mathbf{j}_1)$ to $\phi(\mathbf{j}_2)$.
Now for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ and $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ we have

$$\Omega(\mathrm{id}_{\mathbf{c}})_{\mathbf{j}} = \phi(\mathbf{j})(\mathrm{id}_{\mathbf{c}}) = \mathrm{id}_{\phi(\mathbf{j})(\mathbf{c})} = \mathrm{id}_{\psi_{\mathbf{c}}(\mathbf{j})} \tag{A.73}$$

thus

$$\Omega(\mathrm{id}_{\mathbf{c}}) = \mathrm{id}_{\psi_{\mathbf{c}}} \tag{A.74}$$

and

$$\Lambda(\mathrm{id}_{\mathbf{c}}) = \mathrm{id}_{\Lambda(\mathbf{c})}\,. \tag{A.75}$$

For composable arrows g and f in $\mathcal{M}(\mathbf{C})$ and $\mathbf{j} \in \mathcal{O}(\mathbf{J})$

$$\Omega(g \circ f)_{\mathbf{j}} = \phi(\mathbf{j})(g \circ f) = \phi(\mathbf{j})(g) \circ \phi(\mathbf{j})(f) = \Omega(g)_{\mathbf{j}} \circ \Omega(f)_{\mathbf{j}} \tag{A.76}$$

so

$$\Omega(g \circ f) = \Omega(g) \circ \Omega(f) \tag{A.77}$$

and

$$\Lambda(g \circ f) = \Lambda(g) \circ \Lambda(f). \tag{A.78}$$

Now let us show that for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\gamma_{\mathbf{j}} : \Lambda \to \phi(\mathbf{j})$ is a natural transformation.
Let $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{O}(\mathbf{C})$ and $h : \mathbf{c}_1 \to \mathbf{c}_2$. Then the diagram

$$\begin{array}{ccc} \Lambda(\mathbf{c}_1) & \xrightarrow{(\gamma_{\mathbf{j}})_{\mathbf{c}_1}} & \phi(\mathbf{j})(\mathbf{c}_1) \\ {\scriptstyle \Lambda(f)}\downarrow & & \downarrow{\scriptstyle \phi(\mathbf{j})(f)} \\ \Lambda(\mathbf{c}_2) & \xrightarrow{(\gamma_{\mathbf{j}})_{\mathbf{c}_2}} & \phi(\mathbf{j})(\mathbf{c}_2) \end{array} \tag{A.79}$$

commutes because this is just the definition of $\Lambda(f)$ as inverse limit of $\Omega(f)$.
Now let us show that $\gamma : \mathrm{C}_{\Lambda}^{\mathbf{C},\mathbf{D}^{\mathbf{C}}} \to \phi$ is a natural transformation. For $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{O}(\mathbf{J})$ and $h : \mathbf{j}_1 \to \mathbf{j}_2$ the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\gamma_{\mathbf{j}_1}} & \phi(\mathbf{j}_1) \\ & \searrow^{\gamma_{\mathbf{j}_2}} & \downarrow \phi(h) \\ & & \phi(\mathbf{j}_2) \end{array} \tag{A.80}$$

commutes because for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ this diagram specifies as

$$\begin{array}{ccc} \Lambda(\mathbf{c}) & \xrightarrow{(\gamma_{\mathbf{j}_1})_{\mathbf{c}}} & \phi(\mathbf{j}_1)(\mathbf{c}) \\ & \searrow^{(\gamma_{\mathbf{j}_2})_{\mathbf{c}}} & \downarrow \phi(h)_{\mathbf{c}} \\ & & \phi(\mathbf{j}_2)(\mathbf{c}) \end{array} \tag{A.81}$$

which is by definition of $\gamma_{\mathbf{j}}$

$$\begin{array}{ccc} \Lambda(\mathbf{c}) & \xrightarrow{(\pi_{\mathbf{c}})_{\mathbf{j}_1}} & \psi_{\mathbf{c}}(\mathbf{j}_1) \\ & \searrow^{(\pi_{\mathbf{c}})_{\mathbf{j}_2}} & \downarrow \psi_{\mathbf{c}}(h) \\ & & \psi_{\mathbf{c}}(\mathbf{j}_1) \end{array} \tag{A.82}$$

So (Λ, γ) is a target for ϕ. Let (Λ', γ') be another target for ϕ. Then

$$\gamma' : \mathrm{C}_{\Lambda'}^{\mathbf{J},\mathbf{D}^{\mathbf{C}}} \to \phi \tag{A.83}$$

is a natural transformation, and for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$

$$\gamma'_{\mathbf{j}} : \Lambda' \to \phi(\mathbf{j}) \tag{A.84}$$

is also a natural transformation. Define for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\pi'_{\mathbf{c}} : \mathrm{C}_{\Lambda'(\mathbf{c})}^{\mathbf{J},\mathbf{D}} \to \psi_{\mathbf{c}}$ by $(\pi'_{\mathbf{c}})_{\mathbf{j}} = (\gamma'_{\mathbf{j}})_{\mathbf{c}}$. Then $\pi'_{\mathbf{c}}$ id a natural transformation. In fact, for $f : \mathbf{j}_1 \to \mathbf{j}_2$ the diagram

$$\begin{array}{ccc} \Lambda'(\mathbf{c}) & \xrightarrow{(\pi'_{\mathbf{c}})_{\mathbf{j}_1}} & \psi_{\mathbf{c}}(\mathbf{j}_1) \\ & \searrow^{(\pi'_{\mathbf{c}})_{\mathbf{j}_2}} & \downarrow \psi_{\mathbf{c}}(f) \\ & & \psi_{\mathbf{c}}(\mathbf{j}_2) \end{array} \tag{A.85}$$

commutes because it can be rewritten as

$$\begin{array}{ccc} \Lambda'(\mathbf{c}) & \xrightarrow{(\gamma'_{\mathbf{j}_1})_{\mathbf{c}}} & \phi(\mathbf{j}_1)(\mathbf{c}) \\ & \searrow^{(\gamma'_{\mathbf{j}_2})_{\mathbf{c}}} & \downarrow \phi(f)_{\mathbf{c}} \\ & & \phi(\mathbf{j}_2)(\mathbf{c}) \end{array} \tag{A.86}$$

and

$$\begin{array}{ccc} \Lambda' & \xrightarrow{\gamma'_{\mathbf{j}_1}} & \phi(\mathbf{j}_1) \\ & \underset{\gamma'_{\mathbf{j}_2}}{\searrow} & \downarrow{\scriptstyle \phi(f)} \\ & & \phi(\mathbf{j}_2) \end{array} \tag{A.87}$$

commutes.

Thus for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ there is a unique $\eta_{\mathbf{c}} : \Lambda'(\mathbf{c}) \to \Lambda(\mathbf{c})$ such that the diagram

$$\begin{array}{ccc} \mathrm{C}^{\mathbf{J},\mathbf{D}}_{\Lambda'(\mathbf{c})} & \xrightarrow{\pi'_{\mathbf{c}}} & \psi_{\mathbf{c}} \\ {\scriptstyle \gamma^{\mathbf{J},\,\mathbf{D}}_{\eta_{\mathbf{c}}}}\downarrow & \overset{\pi_{\mathbf{c}}}{\nearrow} & \\ \mathrm{C}^{\mathbf{J},\mathbf{D}}_{\Lambda(\mathbf{c})} & & \end{array} \tag{A.88}$$

commutes, that is the diagram

$$\begin{array}{ccc} \Lambda'(\mathbf{c}) & \xrightarrow{(\pi'_{\mathbf{c}})_{\mathbf{j}}} & \psi_{\mathbf{c}}(\mathbf{j}) \\ {\scriptstyle \eta_{\mathbf{c}}}\downarrow & \overset{(\pi_{\mathbf{c}})_{\mathbf{j}}}{\nearrow} & \\ \Lambda(\mathbf{c}) & & \end{array} \tag{A.89}$$

commutes for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$, which can be rewritten as

$$\begin{array}{ccc} \Lambda'(\mathbf{c}) & \xrightarrow{(\gamma'_{\mathbf{j}})_{\mathbf{c}}} & \phi(\mathbf{j})(\mathbf{c}) \\ {\scriptstyle \eta_{\mathbf{c}}}\downarrow & \overset{(\gamma_{\mathbf{j}})_{\mathbf{c}}}{\nearrow} & \\ \Lambda(\mathbf{c}) & & \end{array} \tag{A.90}$$

Let's prove that the $\eta_{\mathbf{c}}$ define a natural transformation $\eta : \Lambda' \to \Lambda$. Let $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{O}(\mathbf{C})$ and $f : \mathbf{c}_1 \to \mathbf{c}_2$. Since each $\gamma_{\mathbf{j}}$ is a natural transformation we have that for $\mathbf{j} \in \mathcal{O}\,\mathbf{J}$ the diagram

$$\begin{array}{ccc} \Lambda'(\mathbf{c}_1) & \xrightarrow{(\gamma'_{\mathbf{j}})_{\mathbf{c}_1}} & \phi(\mathbf{j})(\mathbf{c}_1) \\ {\scriptstyle \Lambda'(f)}\downarrow & & \downarrow{\scriptstyle \phi(\mathbf{j})} \\ \Lambda'(\mathbf{c}_2) & \xrightarrow{(\gamma'_{\mathbf{j}})_{\mathbf{c}_2}} & \phi(\mathbf{j})(\mathbf{c}_2) \end{array} \tag{A.91}$$

commutes, which can be rewritten as

$$\begin{array}{ccc} \Lambda'(\mathbf{c}_1) & \xrightarrow{(\pi'_{\mathbf{c}_1})_{\mathbf{j}}} & \psi_{\mathbf{c}_1}(\mathbf{j}) \\ {\scriptstyle \Lambda'(f)}\downarrow & & \downarrow{\scriptstyle \Omega(f)_{\mathbf{j}}} \\ \Lambda'(\mathbf{c}_2) & \xrightarrow{(\pi'_{\mathbf{c}_2})_{\mathbf{j}}} & \psi_{\mathbf{c}_2}(\mathbf{j}) \end{array} \tag{A.92}$$

thus also the diagram of natural transformations

$$\begin{array}{ccc} C^{\mathbf{J},\mathbf{D}}_{\Lambda'(\mathbf{c}_1)} & \xrightarrow{\pi'_{\mathbf{c}_1}} & \psi_{\mathbf{c}_1} \\ {\scriptstyle \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda'(f)}}\downarrow & & \downarrow{\scriptstyle \Omega(f)} \\ C^{\mathbf{J},\mathbf{D}}_{\Lambda'(\mathbf{c}_2)} & \xrightarrow{\pi'_{\mathbf{c}_2}} & \psi_{\mathbf{c}_2} \end{array} \tag{A.93}$$

commutes. Set

$$\varepsilon = \Omega(f) \circ \pi'_{\mathbf{c}_1} = \pi'_{\mathbf{c}_2} \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda'(f)}. \tag{A.94}$$

Then $\varepsilon : C^{\mathbf{J},\mathbf{D}}_{\Lambda'(\mathbf{c}_1)} \to \psi_{\mathbf{c}_2}$, so there is a unique natural transformation $\sigma : C^{\mathbf{J},\mathbf{D}}_{\Lambda'(\mathbf{c}_1)} \to C^{\mathbf{J},\mathbf{D}}_{\Lambda(\mathbf{c}_2)}$ such that $\varepsilon = \pi_{\mathbf{c}_2} \circ \sigma$. But

$$\pi_{\mathbf{c}_2} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\eta_{\mathbf{c}_2}} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda'(f)} = \pi'_{\mathbf{c}_2} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda'(f)} = \varepsilon \tag{A.95}$$

and, since $\Lambda(f)$ is a limit of $\Omega(f)$

$$\pi_{\mathbf{c}_2} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda(f)} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\eta_{\mathbf{c}_1}} = \Omega(f) \pi_{\mathbf{c}_1} \gamma^{\mathbf{J},\mathbf{D}}_{\eta_{\mathbf{c}_1}} = \Omega(f) \pi'_{\mathbf{c}_1} = \varepsilon \tag{A.96}$$

thus

$$\gamma^{\mathbf{J},\mathbf{D}}_{\eta_{\mathbf{c}_2}} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda'(f)} = \gamma^{\mathbf{J},\mathbf{D}}_{\Lambda(f)} \circ \gamma^{\mathbf{J},\mathbf{D}}_{\eta_{\mathbf{c}_1}} \tag{A.97}$$

that is the diagram

$$\begin{array}{ccc} \Lambda'(\mathbf{c}_1) & \xrightarrow{\eta_{\mathbf{c}_1}} & \Lambda(\mathbf{c}_1) \\ {\scriptstyle \Lambda'(f)}\downarrow & & \downarrow{\scriptstyle \Lambda(f)} \\ \Lambda'(\mathbf{c}_2) & \xrightarrow{\eta_{\mathbf{c}_2}} & \Lambda(\mathbf{c}_2) \end{array} \tag{A.98}$$

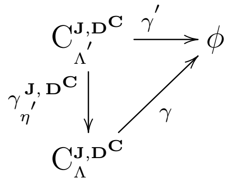
commutes, and η is a natural transformation.
Thus for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ the diagram

$$\begin{array}{ccc} \Lambda' & \xrightarrow{\gamma'_{\mathbf{j}}} & \phi(\mathbf{j}) \\ {\scriptstyle \eta}\downarrow & \nearrow{\scriptstyle \gamma_{\mathbf{j}}} & \\ \Lambda & & \end{array} \tag{A.99}$$

commutes, and so does

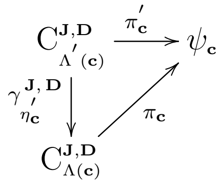
$$\begin{array}{ccc} C^{\mathbf{J},\mathbf{D}^{\mathbf{C}}}_{\Lambda'} & \xrightarrow{\gamma'} & \phi \\ {\scriptstyle \gamma^{\mathbf{J},\mathbf{D}^{\mathbf{C}}}_{\eta}}\downarrow & \nearrow{\scriptstyle \gamma} & \\ C^{\mathbf{J},\mathbf{D}^{\mathbf{C}}}_{\Lambda} & & \end{array} \tag{A.100}$$

Now suppose $\eta : \Lambda' \to \Lambda$ is also such as the diagram



(A.101)

commutes. Then for $\mathbf{c} \in \mathcal{O}\,\mathbf{C}$ the diagram



(A.102)

commutes, and since $\eta_{\mathbf{c}}$ is uniquely determined this yelds $\eta'_{\mathbf{c}} = \eta_{\mathbf{c}}$ for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$, and so $\eta' = \eta$.

Now suppose (Λ, γ) is an inverse limit of ϕ. For $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ the functor $\psi_{\mathbf{c}}$ has a limit, so we can construct a limit (Λ', γ') of ϕ such that $(\Lambda'(\mathbf{c}), \pi'_{\mathbf{c}})$ is a limit of $\psi_{\mathbf{c}}$. Then there is a natural isomorphism $i : \Lambda' \to \Lambda$ such that $\gamma' = \gamma \circ \gamma_i^{\mathbf{J},\mathbf{D}^{\mathbf{C}}}$, and so $\pi'_{\mathbf{c}} = \pi_{\mathbf{c}} \circ \gamma_{i_{\mathbf{c}}}^{\mathbf{J},\mathbf{D}}$, and $(\Lambda(\mathbf{c}), \pi_{\mathbf{c}})$ is a limit of $\psi_{\mathbf{c}}$. ✠

Proposition A.2.40. *Let* $\mathbf{C}$ *be a category. For every* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *the hom-functor* $\overline{\mathbf{C}}(\mathbf{c}, -)$ *preserves inverse limits.*

Proof. Let $\mathrm{F} : \mathbf{J} \to \mathbf{C}$ and $(\mathbf{l}, \lambda)$ an inverse limit of F. Let's show that $\big(\overline{\mathbf{C}}(\mathbf{c}, \mathbf{l}), \overline{\mathbf{C}}(\mathbf{c}, \mu)\big)$ is an inverse limit of $\overline{\mathbf{C}}(\mathbf{c}, \mathrm{F}-)$.

Of course $\big(\overline{\mathbf{C}}(\mathbf{c}, \mathbf{l}), \overline{\mathbf{C}}(\mathbf{c}, \mu)\big)$ is an inverse target of $\overline{\mathbf{C}}(\mathbf{c}, \mathrm{F}-)$.

Let (X, τ) be an inverse target of $\overline{\mathbf{C}}(\mathbf{c}, \mathrm{F}-)$, so $\tau : \mathrm{C}_X^{\mathbf{J},\mathbf{Set}} \to \overline{\mathbf{C}}(\mathbf{c}, \mathrm{F}-)$. If $f \in \overline{\mathbf{J}}(\mathbf{i}, \mathbf{j})$ then $\tau_{\mathbf{j}} = \overline{\mathbf{C}}(\mathbf{c}, \mathrm{F}(f))\tau_{\mathbf{i}}$, so for $x \in X$ $\tau_{\mathbf{j}}(x) = \mathrm{F}(f)\tau_{\mathbf{i}}(x)$, which implies that the $\tau_{\mathbf{j}}(x)$ are the components of a natural transformation $\tau^x : \mathrm{C}_{\mathbf{c}}^{\mathbf{J},\mathbf{C}} \to \mathrm{F}$, and $(\mathbf{c}, \tau^x)$ is an inverse target of F. Thus there is a unique morphism $f^x : \mathbf{c} \to \mathbf{l}$ such that $\tau^x = \mu\gamma_{f^x}^{\mathbf{J},\mathbf{C}}$. The morphisms f^x define a morphism $f : X \to \overline{\mathbf{C}}(\mathbf{c}, \mathbf{l})$ by $f(x) = f^x$, and for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ we have $\tau_{\mathbf{j}} = \overline{\mathbf{C}}(\mathbf{c}, \mu_{\mathbf{j}})f$ and finally $\tau = \overline{\mathbf{C}}(\mathbf{c}, \mu)\gamma_f^{\mathbf{J},\mathbf{Set}}$.

If $g : X \to \overline{\mathbf{C}}(\mathbf{c}, \mathbf{l})$ also is such that $\tau = \overline{\mathbf{C}}(\mathbf{c}, \mu)\gamma_g^{\mathbf{J},\mathbf{Set}}$ then for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ we have $\tau_{\mathbf{j}} = \mu_{\mathbf{j}}g$ and for $x \in X$ $\tau_{\mathbf{j}}(x) = \mu_{\mathbf{j}}g(x)$, so $g = f$. ✠

Lemma A.2.8. *If a functor* F *preserves inverse limits and* G *is naturally isomorphic to* F *then* G *also preserves inverse limits.*

Proof. Let $\mathrm{F} : \mathbf{C} \to \mathbf{D}$, $\mathrm{H} : \mathbf{J} \to \mathbf{C}$, $(\mathbf{l}, \lambda)$ a limit of H, suppose that F preserves inverse limits and that $\phi : \mathrm{F} \to \mathrm{G}$ is a natural isomorphism.

Let $(\mathbf{m}, \mu)$ be an invesre target of GH. Then for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\phi^{-1}_{\mathrm{H(i)}}\mu_{\mathrm{i}} \in \overline{\mathbf{D}}(\mathbf{m}, \mathrm{FH}(\mathbf{i}))$, thus, since F preserves inverse limits, there is a unique morphism $f \in \overline{\mathbf{D}}(\mathbf{m}, \mathrm{F}(\mathbf{l}))$ such that

$$\phi^{-1}_{\mathrm{H(i)}}\mu_{\mathrm{i}} = \mathrm{F}\lambda_{\mathrm{i}} f. \tag{A.103}$$

Set $g = \phi_{\mathrm{l}} f$. Then $g \in \overline{\mathbf{D}}(\mathbf{m}, \mathrm{G}(\mathbf{l}))$ and $\mathrm{G}\lambda_{\mathrm{i}} g = \mathrm{G}\lambda_{\mathrm{i}}\phi_{\mathrm{l}} f = \phi_{\mathrm{H(i)}}\mathrm{F}\lambda_{\mathrm{i}} f = \mu_{\mathrm{i}}$, whence $\mu = \mathrm{G}\lambda\gamma^{\mathbf{J},\mathbf{D}}_{g}$.
If $h \in \overline{\mathbf{D}}(\mathbf{m}, \mathrm{G}(\mathbf{l}))$ is also such that $\mu = \mathrm{G}\lambda\gamma^{\mathbf{J},\mathbf{D}}_{h}$, then for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\mathrm{F}\lambda_{\mathrm{j}}\phi^{-1}_{\mathrm{l}} h = \phi^{-1}_{\mathrm{H(i)}}\mathrm{G}\lambda_{\mathrm{j}} h = \phi^{-1}_{\mathrm{H(j)}}\mu_{\mathrm{j}}$. Since f is the unique morphism satisfying Eq. A.103, $\phi^{-1}_{\mathrm{l}} h = f$ and $h = \phi_{\mathrm{l}} f = g$. ✠

Corollary A.2.4. *A representable functor preserves inverse limits.*

Proof. Straightforward from Proposition A.2.40, Lemma A.2.8, and the definition of representable functor. ✠

Theorem A.2.3 (Inverse-completeness of **Set**)**.** *If* $\mathbf{J}$ *is a small category then every functor from* $\mathbf{J}$ *to* **Set** *has an inverse limit* $(\mathbf{l}^{\mathrm{F}}, \lambda^{\mathrm{F}})$ *where*

$$\mathbf{l}^{\mathrm{F}} = \overline{\mathbf{Set}^{\mathbf{J}}}(\mathrm{C}^{\mathbf{J},\mathbf{Set}}_{\{0\}}, \mathrm{F}) \tag{A.104}$$

and for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$, $\tau \in \mathbf{l}^{\mathrm{F}}$

$$\lambda^{\mathrm{F}}_{\mathrm{j}}(\tau) = \tau_{\mathrm{j}}(0). \tag{A.105}$$

Proof. Since $\mathbf{J}$ is a small category $\overline{\mathbf{Set}^{\mathbf{J}}}(\mathrm{C}^{\mathbf{J},\mathbf{Set}}_{\{0\}}, \mathrm{F}) \in \mathcal{O}(\mathbf{Set})$.
Let's show that λ^{F} is a natural transformation. If $f \in \overline{\mathbf{J}}(\mathbf{i}, \mathbf{j})$ then for $\tau \in \mathbf{l}^{\mathrm{F}}$, since τ is a natural transformation

$$\lambda^{\mathrm{F}}_{\mathrm{j}}(\tau) = \tau_{\mathrm{j}}(0) = \mathrm{F}(f)\tau_{\mathrm{i}}(0) = \mathrm{F}(f)\lambda^{\mathrm{F}}_{\mathrm{i}}(\tau) \tag{A.106}$$

thus $\lambda^{\mathrm{F}}_{\mathrm{j}} = \mathrm{F}(f)\lambda^{\mathrm{F}}_{\mathrm{i}}$.
Let's show that $(\mathbf{l}^{\mathrm{F}}, \lambda^{\mathrm{F}})$ is a universal arrow.
If $\sigma \in \overline{\mathbf{Set}^{\mathbf{J}}}(\mathrm{C}^{\mathbf{J},\mathbf{Set}}_{X}, \mathrm{F})$ let for $x \in \mathrm{X}$ $\sigma^x \in \overline{\mathbf{Set}^{\mathbf{J}}}\left(\mathrm{C}^{\mathbf{J},\mathbf{Set}}_{\{0\}}, \mathrm{F}\right)$ defined for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ by $\sigma^x_{\mathrm{j}}(0) = \sigma_{\mathrm{j}}(x)$.
Tus we have a map

$$\begin{aligned} h : \mathrm{X} &\to \overline{\mathbf{Set}^{\mathbf{J}}}(\mathrm{C}^{\mathbf{J},\mathbf{Set}}_{\{0\}}, \mathrm{F}) \\ x &\mapsto \sigma^x \end{aligned} \tag{A.107}$$

such that for $x \in \mathrm{X}$ and $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\lambda^{\mathrm{F}}_{\mathrm{j}}(h(x)) = \lambda^{\mathrm{F}}_{\mathrm{j}}(\sigma^x) = \sigma^x_{\mathrm{j}}(0) = \sigma_{\mathrm{j}}(x)$, thus $\lambda^{\mathrm{F}}_{\mathrm{j}} h = \sigma_{\mathrm{j}}$.
If also $k : \mathrm{X} \to \overline{\mathbf{Set}^{\mathbf{J}}}(\mathrm{C}^{\mathbf{J},\mathbf{Set}}_{\{0\}}, \mathrm{F})$ is such that $\lambda^{\mathrm{F}}_{\mathrm{j}} k = \sigma_{\mathrm{j}}$ for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$, then for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ and $x \in \mathrm{X}$ $\lambda^{\mathrm{F}}_{\mathrm{j}}(k(x)) = k(x)_{\mathrm{j}}(0) = \sigma_{\mathrm{j}}(x) = \sigma^x_{\mathrm{j}}(0)$, thus $k(x) = \sigma^x$ and $k = h$. ✠

A.3 Pullbacks and pushouts

Definition A.3.1. A ***pulllback*** is a limit of a functor $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ where

- $\mathcal{O}(\mathbf{C}) = \{\mathbf{c}_1, \ldots, \mathbf{c}_n, \mathbf{d}\}$
- $\mathcal{M}(\mathbf{C}) = \{f_1, \ldots, f_n\}$ where $f_i : \mathbf{c}_i \to \mathbf{d}$.

Remark A.3.1. If $(\mathbf{p}, \pi)$ is a pullback of $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ then for $i = 1, \ldots, n$ $\pi_{\mathbf{d}} = \mathrm{F}(f_i) \circ \pi_{\mathbf{c}_i}$, so a pullback equalises all the morphisms $\mathrm{F}(f_i) \circ \pi_{\mathbf{c}_i}$. If $(\mathbf{p}', \pi')$ is also a pullback of F then there is a unique $h : \mathbf{p}' \to \mathbf{p}$ such that $\pi' = \pi \circ \gamma_h^{\mathbf{C},\mathbf{D}}$.
It can be useful to refer to a pullback of a finite set of morphisms. If for $i = 1, \ldots, n$ $f_i : \mathbf{c}_i \to \mathbf{d}$ are morphisms of a category $\mathbf{C}$ with a common codomain, then $(\mathbf{p}, \pi_1, \ldots, \pi_n)$ is a pullback of $f_1, \ldots, f_n$ if

- $f_i \circ \pi_i = f_j \circ \pi_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, n$
- if $(\mathbf{p}', \pi_1, \ldots, \pi_n')$ is such that $f_i \circ \pi_i' = f_j \circ \pi_j'$ for $i = 1, \ldots, n$, $j = 1, \ldots, n$ then there is a unique $h : \mathbf{p}' \to \mathbf{p}$ such that $\pi_i' = \pi_i \circ h$ for $i = 1, \ldots, n$.

Likewise, it can be useful to say that a diagram like

$$\begin{array}{ccc} \mathbf{p} & \xrightarrow{h} & \mathbf{b} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{c} & \xrightarrow{g} & \mathbf{d} \end{array} \tag{A.108}$$

is a pullback, meaning that $(\mathbf{p}, h, k)$ is a pullback of f, g.

Lemma A.3.1. *A morphism $f : \mathbf{a} \to \mathbf{b}$ is a monomorphism if and only if the diagram*

$$\begin{array}{ccc} \mathbf{a} & \xrightarrow{\mathrm{id}_{\mathbf{a}}} & \mathbf{a} \\ {\scriptstyle \mathrm{id}_{\mathbf{a}}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{a} & \xrightarrow{f} & \mathbf{b} \end{array} \tag{A.109}$$

is a pullback.

Proof. Just a check. ✠

Proposition A.3.1. *If a category has binary inverse products and inverse equalisers then it has pullbacks.*

Proof. Let $f_1 : \mathbf{a}_1 \to \mathbf{b}$, $f_2 : \mathbf{a}_2 \to \mathbf{b}$, p_1 and p_2 the projections from $\mathbf{a}_1 \Pi \mathbf{a}_2$, and $k : \mathbf{e} \to \mathbf{a}_1 \Pi \mathbf{a}_2$ an inverse equaliser of $f_1 p_1, f_2 p_2$. Let's show that $(\mathbf{e}, p_1 k, p_2 k)$ is a pullback of f_1, f_2. Of course $f_1 p_1 k = f_2 p_2 k$. If $h_1 : \mathbf{d} \to \mathbf{a}_1$, $h_2 : \mathbf{d} \to \mathbf{a}_2$ are such that $f_1 h_1 = f_2 h_2$ then there is $h' : \mathbf{d} \to \mathbf{a}_1 \Pi \mathbf{a}_2$ such that $h_1 = p_1 h'$, $h_2 = p_2 h'$ and $f_1 p_1 h' = f_1 h_1 = f_2 h_2 = f_2 p_2 h'$ so there is $h : \mathbf{d} \to \mathbf{e}$ such that $h' = kh$ and thus $h_1 = p_1 kh$, $h_2 = p_2 kh$.
If $h^* : \mathbf{d} \to \mathbf{e}$ is also such that $h_1 = p_1 kh^*$, $h_2 = p_2 kh^*$, then $h' = kh^*$, because h' is uniquely determined, and $h^* = h$ because the factorisation of h' through k is unique. ✠

Proposition A.3.2. *If a category* $\mathbf{C}$ *has binary pullbacks then every natural transformation* τ *in* $\mathbf{C}$ *is a monomorphism if and only if* $\tau_{\mathbf{c}}$ *is a monomorphism for every* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$.

Proof. Let $\mathrm{F}, \mathrm{G} : \mathbf{B} \to \mathbf{C}$, and $\tau : \mathrm{F} \to \mathrm{G}$. By Theorem A.2.2 $\mathbf{C}^{\mathbf{B}}$ has binary pullbacks, and by Lemma A.3.1 the diagram

$$\begin{array}{ccc} \mathrm{F} & \xrightarrow{\mathrm{id}_{\mathrm{F}}} & \mathrm{F} \\ {\scriptstyle \mathrm{id}_{\mathrm{F}}}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathrm{F} & \xrightarrow{\tau} & \mathrm{G} \end{array} \tag{A.110}$$

is a pullback in $\mathbf{C}^{\mathbf{B}}$. Then again by Theorem A.2.2 for every $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ the diagram

$$\begin{array}{ccc} \mathrm{F}(\mathbf{c}) & \xrightarrow{(\mathrm{id}_{\mathrm{F}})_{\mathbf{c}}} & \mathrm{F}(\mathbf{c}) \\ {\scriptstyle (\mathrm{id}_{\mathrm{F}})_{\mathbf{c}}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathbf{c}}} \\ \mathrm{F}(\mathbf{c}) & \xrightarrow{\tau_{\mathbf{c}}} & \mathrm{G}(\mathbf{c}) \end{array} \tag{A.111}$$

is a pullback in $\mathbf{C}$, so by Lemma A.3.1 $\tau_{\mathbf{c}}$ is a monomorphism.
Conversely, if for every $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $\tau_{\mathbf{c}}$ is a monomorphism, then for every $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ the diagram A.111 is a pullback, and so is diagram A.110 and τ is a monomorphism. ✠

A.4 Preorders

Definition A.4.1. A ***preorder*** $\mathbf{P}$ is a category such that for any pair of objects $\mathbf{p}_1, \mathbf{p}_2$ of $\mathbf{P}$ there is at most one morphism from $\mathbf{p}_1$ to $\mathbf{p}_2$.

Notation A.4.1. If $\mathbf{P}$ is a preorder and $\mathbf{p}_1$, $\mathbf{p}_2$ are objects of $\mathbf{P}$ such that $\overline{\mathbf{P}}(\mathbf{p}_1, \mathbf{p}_2) \neq \varnothing$, we will write

$$\mathbf{p}_1 \preccurlyeq \mathbf{p}_2. \tag{A.112}$$

If $\mathbf{p}_1 \preccurlyeq \mathbf{p}_2$ we will denote with

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle \tag{A.113}$$

the unique element of $\overline{\mathbf{P}}(\mathbf{p}_1, \mathbf{p}_2)$. If $\mathrm{F} : \mathbf{P} \to \mathbf{C}$ is a functor and $\mathbf{p}_1 \preccurlyeq \mathbf{p}_2$ then we will set

$$\mathrm{F}^{\mathbf{p}_1}_{\mathbf{p}_2} = \mathrm{F}(\langle \mathbf{p}_1, \mathbf{p}_2 \rangle). \tag{A.114}$$

Definition A.4.2. A preorder $\mathbf{P}$ is ***directed*** if for any pair of objects $\mathbf{p}_1, \mathbf{p}_2$ of $\mathbf{P}$ there is an object $\mathbf{q}$ of $\mathbf{P}$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}$ and $\mathbf{p}_2 \preccurlyeq \mathbf{q}$.

Lemma A.4.1. *Let* $\mathbf{P}$ *be a preorder,* $\mathrm{F} : \mathbf{P} \to \mathbf{Set}$ *a functor and* $(\mathbf{t}, \tau)$ *a direct target for* F *such that*

(1) $\forall t \in \mathbf{t}\ \exists\, \mathbf{p} \in \mathcal{O}(\mathbf{P}) : t \in \mathfrak{Im}(\tau_{\mathbf{p}})$ *?????*

(2) $\forall \mathbf{p}_1 \in \mathcal{O}(\mathbf{P})\ \forall \mathbf{p}_2 \in \mathcal{O}(\mathbf{P})\ \forall t_1 \in \mathrm{F}(\mathbf{p}_1)\ \forall t_2 \in \mathrm{F}(\mathbf{p}_2)$
$\tau_{\mathbf{p}_1}(t_1) = \tau_{\mathbf{p}_2}(t_2) \iff \exists\, \mathbf{q} \in \mathcal{O}(\mathbf{P}) : \mathbf{p}_1 \preccurlyeq \mathbf{q} \wedge \mathbf{p}_2 \preccurlyeq \mathbf{q} \wedge \mathrm{F}_{\mathbf{q}}^{\mathbf{p}_1}(t_1) = \mathrm{F}_{\mathbf{q}}^{\mathbf{p}_2}(t_2)$.

Then $(\mathbf{t}, \tau)$ *is a direct limit for* F.

Proof. Let $(\mathbf{s}, \sigma)$ be any direct target for F. If $t \in \mathbf{t}$, then by **(1)** $t = \tau_{\mathbf{p}}(s)$ for some object $\mathbf{p}$ of $\mathbf{P}$ and some element s of $\mathrm{F}(\mathbf{p})$. We will show that $\sigma_{\mathbf{p}}(s)$ depends only on t, that is, if $t = \tau_{\mathbf{q}}(u)$ also holds for some object $\mathbf{q}$ of $\mathbf{P}$ and some element u of $\mathrm{F}(\mathbf{q})$, then $\sigma_{\mathbf{q}}(u) = \sigma_{\mathbf{p}}(s)$. Indeed, by **(2)** there is an object $\mathbf{r}$ of $\mathbf{P}$ such that $\mathbf{p} \preccurlyeq \mathbf{r}$, $\mathbf{q} \preccurlyeq \mathbf{r}$ and $\mathrm{F}_{\mathbf{r}}^{\mathbf{p}}(s) = \mathrm{F}_{\mathbf{r}}^{\mathbf{q}}(u)$, so

$$\sigma_{\mathbf{q}}(u) = \sigma_{\mathbf{r}}(\mathrm{F}_{\mathbf{r}}^{\mathbf{q}}(u)) = \sigma_{\mathbf{r}}(\mathrm{F}_{\mathbf{r}}^{\mathbf{p}}(s)) = \sigma_{\mathbf{p}}(s). \tag{A.115}$$

Then we can define a morphism in **Set**

$$\begin{aligned} f : \mathbf{t} &\to \mathbf{s} \\ t &\mapsto \sigma_{\mathbf{p}}(s) \end{aligned} \tag{A.116}$$

where $\mathbf{p}$ and s are any object of $\mathbf{P}$ and any element of $\mathrm{F}(\mathbf{p})$ such that $t = \tau_p(s)$. If $\mathbf{p}$ is any object of $\mathbf{P}$ then by the very definition of f we have $\sigma_{\mathbf{p}} = f \circ \tau_{\mathbf{p}}$. If $g : \mathbf{t} \to \mathbf{s}$ is any morphism in **Set** such that, for any object $\mathbf{p}$ of $\mathbf{P}$, $\sigma_{\mathbf{p}} = g \circ \tau_{\mathbf{p}}$ holds, then for $t \in \mathbf{t}$ such that $t = \tau_{\mathbf{p}}(s)$ for some object $\mathbf{p}$ of $\mathbf{P}$ and some element s of $\mathrm{F}(\mathbf{p})$ we have $g(t) = g(\tau_{\mathbf{p}}(s)) = \sigma_{\mathbf{p}}(s) = f(t)$, that is, $g = f$. ✠

Proposition A.4.1. *If* $\mathbf{P}$ *is a directed preorder for any functor* $\mathrm{F} : \mathbf{P} \to \mathbf{Set}$ *there is a direct target for* F *which satisfies conditions **(1)** and **(2)** of Lemma A.4.1.*

Proof. Let

$$\mathbf{t}_0 = \bigcup_{\mathbf{p} \in \mathcal{O}(\mathbf{P})} \{\mathbf{p}\} \times \mathrm{F}(\mathbf{p}). \tag{A.117}$$

On the set $\mathbf{t}_o$ define the relation $\sim$ by:

$$(\mathbf{p}_1, s_1) \sim (\mathbf{p}_2, s_2) \iff \exists\, \mathbf{q} \in \mathcal{O}(\mathbf{P}) : \mathbf{p}_1 \preccurlyeq \mathbf{q} \wedge \mathbf{p}_2 \preccurlyeq \mathbf{q} \wedge \mathrm{F}_{\mathbf{q}}^{\mathbf{p}_1}(s_1) = \mathrm{F}_{\mathbf{q}}^{\mathbf{p}_2}(s_2). \tag{A.118}$$

Then $\sim$ is an equivalence relation .
Indeed, it is clearly reflexive and symmetric. Suppose $(\mathbf{p}_1, s_1) \sim (\mathbf{p}_2, s_2)$ and $(\mathbf{p}_2, s_2) \sim (\mathbf{p}_3, s_3)$. Then there are objects $\mathbf{q}_1$ and $\mathbf{q}_2$ of $\mathbf{P}$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}_1$, $\mathbf{p}_2 \preccurlyeq \mathbf{q}_1$, $\mathbf{p}_2 \preccurlyeq \mathbf{q}_2$, $\mathbf{p}_3 \preccurlyeq \mathbf{q}_2$ and $\mathrm{F}_{\mathbf{q}_1}^{\mathbf{p}_1}(s_1) = \mathrm{F}_{\mathbf{q}_1}^{\mathbf{p}_2}(s_2)$ and $\mathrm{F}_{\mathbf{q}_2}^{\mathbf{p}_2}(s_2) = \mathrm{F}_{\mathbf{q}_2}^{\mathbf{p}_3}(s_3)$. But since $\mathbf{P}$ is directed there is $\mathbf{r}$ such that $\mathbf{q}_1 \preccurlyeq \mathbf{r}$ and $\mathbf{q}_2 \preccurlyeq \mathbf{r}$, and $\mathrm{F}_{\mathbf{r}}^{\mathbf{p}_1}(s_1) = \mathrm{F}_{\mathbf{r}}^{\mathbf{q}_1}(\mathrm{F}_{\mathbf{q}_1}^{\mathbf{p}_1}(s_1)) = \mathrm{F}_{\mathbf{r}}^{\mathbf{q}_1}(\mathrm{F}_{\mathbf{q}_1}^{\mathbf{p}_2}(s_2)) = \mathrm{F}_{\mathbf{r}}^{\mathbf{p}_2}(s_2) = \mathrm{F}_{\mathbf{r}}^{\mathbf{q}_2}(\mathrm{F}_{\mathbf{q}_2}^{\mathbf{p}_2}(s_2)) = \mathrm{F}_{\mathbf{r}}^{\mathbf{q}_2}(\mathrm{F}_{\mathbf{q}_2}^{\mathbf{p}_3}(s_3)) = \mathrm{F}_{\mathbf{r}}^{\mathbf{p}_3}(s_3)$, so $\sim$ is transitive.
Let $\mathbf{t} = \mathbf{t}_0/\sim$. For $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ set

$$\begin{aligned} \tau_{\mathbf{p}} : \mathrm{F}(\mathbf{p}) &\to \mathbf{t} \\ s &\mapsto [(\mathbf{p}, s)]\,. \end{aligned} \tag{A.119}$$

If $\mathbf{p}, \mathbf{q}$ are objects of $\mathbf{P}$ and $\mathbf{p} \preccurlyeq \mathbf{q}$, and $s \in \mathbf{p}$, then

$$\tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s)) = \big[(\mathbf{q}, \mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s))\big]$$

and

$$\tau_{\mathbf{p}}(s) = [(\mathbf{p}, s)]\,.$$

But $\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s) = \mathrm{F}^{\mathbf{q}}_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s))$ whence $\big[(\mathbf{q}, \mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s))\big] = [(\mathbf{p}, s)]$ and $\tau_{\mathbf{p}} = \tau_{\mathbf{q}} \circ \mathrm{F}^{\mathbf{p}}_{\mathbf{q}}$, so $(\mathbf{t}, \tau)$ is a direct target for F.

If $t \in \mathbf{t}$ then $t = [(\mathbf{p}, s)]$ for some $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and some $s \in \mathrm{F}(\mathbf{p})$, so $t = \tau_{\mathbf{p}}(s)$.

If $\tau_{\mathbf{p}}(s) = \tau_{\mathbf{q}}(u)$ for some objects $\mathbf{p}, \mathbf{q}$ of $\mathbf{P}$ and some $s \in \mathrm{F}(\mathbf{p})$ and $u \in \mathrm{F}(\mathbf{q})$, then $[(\mathbf{p}, s)] = [(\mathbf{q}, u)]$, which means that there is $\mathbf{r} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p} \preccurlyeq \mathbf{r}$, $\mathbf{q} \preccurlyeq \mathbf{r}$ and $\mathrm{F}^{\mathbf{p}}_{\mathbf{r}}(s) = \mathrm{F}^{\mathbf{q}}_{\mathbf{r}}(u)$. ✠

Corollary A.4.1. *If* $\mathbf{P}$ *is a directed preorder every functor* $\mathrm{F}\colon\mathbf{P}\to\mathbf{Set}$ *has a direct limit.*

Proposition A.4.2. *If* $\mathbf{P}$ *is a directed preorder, a direct target* $(\mathbf{t}, \tau)$ *for the functor* $\mathrm{F}\colon\mathbf{P}\to\mathbf{Set}$ *is a direct limit if and only if it satisfies conditions **(1)** and **(2)** of Lemma A.4.1.*

Proof. In Lemma A.4.1 was already proved that a direct target $(\mathbf{t}, \tau)$ for F satisfying **(1)** and **(2)** is a direct limit for F.

Suppose that $(\mathbf{t}, \tau)$ is a direct limit for F. By Proposition A.4.1 there is a direct limit $(\mathbf{l}, \lambda)$ for F which satisfies **(1)** and **(2)**. Let $f : \mathbf{l} \to \mathbf{t}$ be the unique bijection such that for every $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ $\tau_{\mathbf{p}} = f \circ \lambda_{\mathbf{p}}$.

Let $t \in \mathbf{t}$. Then $t = f(l)$ for some $l \in \mathbf{l}$, and $l = \lambda_{\mathbf{p}}(s)$ for some $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and some $s \in F(p)$, so $t = \tau_{\mathbf{p}}(s)$.

Let $\mathbf{p} \in \mathcal{O}(\mathbf{P})$, $\mathbf{q} \in \mathcal{O}(\mathbf{P})$, $u \in \mathrm{F}(\mathbf{p})$, $v \in \mathrm{F}(\mathbf{q})$ such that $\tau_{\mathbf{p}}(u) = \tau_{\mathbf{q}}(v)$; then $f(\lambda_{\mathbf{p}}(u)) = f(\lambda_{\mathbf{p}}(v))$ whence $\lambda_{\mathbf{p}}(u) = \lambda_{\mathbf{p}}(v)$; so there is $\mathbf{r} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p} \preccurlyeq \mathbf{r}$, $\mathbf{q} \preccurlyeq \mathbf{r}$ and $\mathrm{F}^{\mathbf{p}}_{\mathbf{r}}(u) = \mathrm{F}^{\mathbf{q}}_{\mathbf{r}}(v)$. ✠

Definition A.4.3. Let $\mathbf{P}$ be a directed preorder, $\mathrm{F}\colon\mathbf{P}\to\mathbf{Set}$ a functor. We call the ***standard direct limit of*** F the direct limit constructed as in Proposition A.4.1 by (A.117), (A.118) and (A.119).

Proposition A.4.3. *The forgetful functor* $\mathrm{U}\colon\mathbf{Grp}\to\mathbf{Set}$ *creates direct limits for any functor from a directed preorder.*

Proof. Let $\mathbf{P}$ be a directed preorder, $\mathrm{F}\colon\mathbf{P}\to\mathbf{Grp}$ and $\mathrm{H} = \mathrm{U} \circ \mathrm{F}$. Let $(\mathbf{s}, \tau) \in \varinjlim \mathrm{H}$. For x_1 and x_2 in $\mathbf{s}$ let $x_1 = \tau_{\mathbf{p}_1}(s_1)$, $x_2 = \tau_{\mathbf{p}_2}(s_2)$ for $\mathbf{p}_1 \in \mathcal{O}(\mathbf{P})$, $\mathbf{p}_2 \in \mathcal{O}(\mathbf{P})$, $s_1 \in \mathrm{F}(\mathbf{p}_1)$, $s_2 \in \mathrm{F}(\mathbf{p}_2)$. If $\mathbf{q} \in \mathcal{O}(\mathbf{P})$ is such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}$ and $\mathbf{p}_2 \preccurlyeq \mathbf{q}$, let's show that $\tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2))$ does not depend on q. Let $\mathbf{q}_1 \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}_1$ and $\mathbf{p}_2 \preccurlyeq \mathbf{q}_1$, and $\mathbf{q}_2 \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}_2$ and $\mathbf{p}_2 \preccurlyeq \mathbf{q}_2$. Then there is $\mathbf{r} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{q}_1 \preccurlyeq \mathbf{r}$ and $\mathbf{q}_2 \preccurlyeq \mathbf{r}$, so

$$\begin{aligned}
\tau_{\mathbf{q}_1}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}_1}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}_1}(s_2)) &= \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{q}_1}_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}_1}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}_1}(s_2))) = \\
&= \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{q}_1}_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}_1}(s_1))\mathrm{F}^{\mathbf{q}_1}_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}_1}(s_2))) = \\
&= \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{r}}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{r}}(s_2)) = \\
&= \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{q}_2}_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}_2}(s_1))\mathrm{F}^{\mathbf{q}_2}_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}_2}(s_2))) = \\
&= \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{q}_2}_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}_2}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}_2}(s_2))) = \\
&= \tau_{\mathbf{q}_2}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}_2}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}_2}(s_2)).
\end{aligned}$$

Let's prove that $\tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2))$ with $\mathbf{p}_1 \preccurlyeq \mathbf{q}$ and $\mathbf{p}_2 \preccurlyeq \mathbf{r}$ does not depend either on $\mathbf{p}_1, \mathbf{p}_2$ and s_1, s_2.
If also $x_1 = \tau_{\mathbf{o}_1}(t_1)$, $x_2 = \tau_{\mathbf{o}_2}(t_2)$ for $\mathbf{o}_1 \in \mathcal{O}(\mathbf{P})$, $\mathbf{o}_2 \in \mathcal{O}(\mathbf{P})$, $t_1 \in \mathrm{F}(\mathbf{o}_1)$, $t_2 \in \mathrm{F}(\mathbf{o}_2)$, then there is $\mathbf{q} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{o}_1 \preccurlyeq \mathbf{q}$, $\mathbf{o}_2 \preccurlyeq \mathbf{q}$, $\mathbf{p}_1 \preccurlyeq \mathbf{q}$, $\mathbf{p}_2 \preccurlyeq \mathbf{q}$ and $\mathrm{F}^{\mathbf{o}_1}_{\mathbf{q}}(t_1) = \mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)$, $\mathrm{F}^{\mathbf{o}_2}_{\mathbf{q}}(t_2) = \mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2)$, so $\tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{o}_1}_{\mathbf{q}}(t_1)\mathrm{F}^{\mathbf{o}_2}_{\mathbf{q}}(t_2)) = \tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2))$.
For x_1 and x_2 in $\mathbf{s}$ we can now define their product $x_1x_2 = \tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2))$ where $\mathbf{p}_1$, $\mathbf{p}_2$ are objects of $\mathbf{P}$ and $s_1 \in \mathrm{F}(\mathbf{p}_1)$, $s_2 \in \mathrm{F}(\mathbf{p}_2)$ such that $x_1 = \tau_{\mathbf{p}_1}(s_1)$ and $x_2 = \tau_{\mathbf{p}_2}(s_2)$, and $\mathbf{q}$ is an object of $\mathbf{P}$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}$ and $\mathbf{p}_2 \preccurlyeq \mathbf{q}$.
If also $x_3 = \tau_{\mathbf{p}_3}(s_3)$ then for $\mathbf{q}$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}$, $\mathbf{p}_2 \preccurlyeq \mathbf{q}$, $\mathbf{p}_3 \preccurlyeq \mathbf{q}$

$$\begin{aligned}
x_1(x_2x_3) &= \tau_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)(\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2)\mathrm{F}^{\mathbf{p}_3}_{\mathbf{q}}(s_3))) = \\
&= \tau_{\mathbf{q}}((\mathrm{F}^{\mathbf{p}_1}_{\mathbf{q}}(s_1)\mathrm{F}^{\mathbf{p}_2}_{\mathbf{q}}(s_2))\mathrm{F}^{\mathbf{p}_3}_{\mathbf{q}}(s_3)) = \\
&= (x_1x_2)x_3.
\end{aligned}$$

If $e_{\mathbf{q}}$ is the unit element of $\mathrm{P}(\mathbf{q})$, then let $x = \tau_{\mathbf{p}}(s)$ and $\mathbf{r} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{q} \preccurlyeq \mathbf{r}$ and $\mathbf{p} \preccurlyeq \mathbf{r}$; we have $\tau_{\mathbf{q}}(e_{\mathbf{q}})x = \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{q}}_{\mathbf{r}}(e_{\mathbf{q}})\mathrm{F}^{\mathbf{p}}_{\mathbf{r}}(s)) = \tau_{\mathbf{r}}(e_{\mathbf{r}}\mathrm{F}^{\mathbf{p}}_{\mathbf{r}}(s)) = \tau_{\mathbf{r}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{r}}(s)) = \tau_{\mathbf{p}}(s) = x$. So $\tau_{\mathbf{q}}(e_{\mathbf{q}})$ is the unit element $e_{\mathbf{s}}$ of $\mathbf{s}$, for any object $\mathbf{q}$ in $\mathbf{P}$.
If $x = \tau_{\mathbf{p}}(s)$, and $e_{\mathbf{p}}$ is the unit element of $\mathrm{P}(\mathbf{p})$, then $x\tau_{\mathbf{p}}(s^{-1}) = \tau_{\mathbf{p}}(ss^{-1}) = \tau_{\mathbf{p}}(e_{\mathbf{p}}) = e_{\mathbf{s}}$ and $\tau_{\mathbf{p}}(s^{-1})x = \tau_{\mathbf{p}}(s^{-1}s) = \tau_{\mathbf{p}}(e_{\mathbf{p}}) = e_{\mathbf{s}}$, so $\tau_{\mathbf{p}}(s^{-1}) = x^{-1}$. ✠

Proposition A.4.4. *The forgetful functor* $\mathrm{U}\colon \mathbf{Alg}_\tau \to \mathbf{Set}$ *creates direct limits for any functor from a directed preorder.*

Proof. Under construction. ✠

Definition A.4.4. Let $\mathbf{P}$ be a directed preorder, $\mathrm{F}\colon \mathbf{P} \to \mathbf{Alg}_\tau$ a functor, $\mathrm{U}\colon \mathbf{Alg}_\tau \to \mathbf{Set}$ the forgetful functor for $\mathbf{Alg}_\tau$. We call the ***standard direct limit of*** F the direct limit $\mathbf{a}$ such that $\mathrm{U}(\mathbf{a})$ is the direct limit of $\mathrm{U} \circ \mathrm{F}$ in $\mathbf{Set}$ constructed as in Proposition A.4.1 by (A.117), (A.118) and (A.119), given the algebraic structure τ as in Proposition A.4.4.

Proposition A.4.5. *If* $\mathbf{P}$ *is a directed preorder, a direct target* $(\mathbf{t}, \tau)$ *for the functor* $\mathrm{F}:\mathbf{P}\to\mathbf{Alg}_\tau$ *is a direct limit if and only if it satisfies conditions* ***(1)*** *and* ***(2)*** *of Lemma A.4.1.*

Proof. Obvious. ✠

Lemma A.4.2. *Let* $\mathbf{P}$ *be a directed preorder,* $\mathrm{F}:\mathbf{P}\to\mathbf{Set}$*. If* $(\mathbf{g}, \gamma)$ *is a direct limit for* F *then:*

1. *for* $y \in \mathbf{g}$ *there are* $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ *and* $s \in \mathrm{F}(\mathbf{p})$ *such that* $y = \gamma_{\mathbf{p}}(s)$*;*
2. *if for* $\mathbf{p}_1 \in \mathcal{O}(\mathbf{P})$*,* $\mathbf{p}_2 \in \mathcal{O}(\mathbf{P})$*,* $s_1 \in \mathrm{F}(\mathbf{s}_1)$*,* $s_2 \in \mathrm{F}(\mathbf{s}_2)$ $\gamma_{\mathbf{p}_1}(s_1) = \gamma_{\mathbf{p}_2}(s_2)$ *then there is* $\mathbf{q} \in \mathcal{O}(\mathbf{P})$ *such that* $\mathbf{p}_1 \preccurlyeq \mathbf{q}$ *and* $\mathbf{p}_2 \preccurlyeq \mathbf{q}$*, and* $\mathrm{F}_{\mathbf{q}}^{\mathbf{p}_1}(s_1) = \mathrm{F}_{\mathbf{q}}^{\mathbf{p}_2}(s_1)$*.*

Proof. Let $(\mathbf{f}, \phi)$ be the standard direct limit of F. There is an isomorphism $i : \mathbf{f} \to \mathbf{g}$ such that for $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ $i \circ \phi_{\mathbf{p}} = \gamma_{\mathbf{p}}$.
Let $x \in \mathbf{f}$ such that $y = i(x)$. Since $(\mathbf{f}, \phi)$ is the standard direct limit of F there are $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and $s \in \mathrm{F}(\mathbf{p})$ such that $x = \phi_{\mathbf{p}}(s)$, thus $y = i(\phi_{\mathbf{p}}(s)) = \gamma_{\mathbf{p}}(s)$.
If $\gamma_{\mathbf{p}_1}(s_1) = \gamma_{\mathbf{p}_2}(s_2)$ then $i(\phi_{\mathbf{p}_1}(s_1)) = i(\phi_{\mathbf{p}_2}(s_2))$, thus $\phi_{\mathbf{p}_1}(s_1) = \phi_{\mathbf{p}_2}(s_2)$. Since $(\mathbf{f}, \phi)$ is the standard direct limit of F there is $\mathbf{q} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{q}$ and $\mathbf{p}_2 \preccurlyeq \mathbf{q}$, and $\mathrm{F}_{\mathbf{q}}^{\mathbf{p}_1}(s_1) = \mathrm{F}_{\mathbf{q}}^{\mathbf{p}_2}(s_1)$. ✠

Proposition A.4.6. *Let* $\mathbf{P}$ *be a directed preorder,* F *and* G *functors from* $\mathbf{P}$ *to* $\mathbf{Set}$*,* τ *a natural transformation from* F *to* G *such that, for each* $\mathbf{p} \in \mathcal{O}(\mathbf{P})$*,* $\tau_{\mathbf{p}}$ *is injective. Then any direct limit of* τ *is injective.*

Proof. Let $\bar{\tau}$ be the direct limit of τ relative to the direct limits $(\mathbf{f}, \phi)$ of F and $(\mathbf{g}, \gamma)$ of G. If $x_1 \in \mathbf{f}$ and $x_2 \in \mathbf{f}$ are such that $\bar{\tau}(x_1) = \bar{\tau}(x_2)$, by Lemma A.4.2 there are $\mathbf{p}_1 \in \mathcal{O}(\mathbf{P})$, $\mathbf{p}_1 \in \mathcal{O}(\mathbf{P})$, $s_1 \in \mathrm{F}(\mathbf{p}_1)$, $s_2 \in \mathrm{F}(\mathbf{p}_2)$ such that $x_1 = \phi_{\mathbf{p}_1}(s_1)$, $x_2 = \phi_{\mathbf{p}_2}(s_2)$, then $\gamma_{\mathbf{p}_1}(\tau_{\mathbf{p}_1}(s_1)) = \gamma_{\mathbf{p}_2}(\tau_{\mathbf{p}_2}(s_2))$ so, again by Lemma A.4.2, there is $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p}_1 \preccurlyeq \mathbf{p}$, $\mathbf{p}_2 \preccurlyeq \mathbf{p}$ and $\mathrm{G}_{\mathbf{p}}^{\mathbf{p}_1}(\tau_{\mathbf{p}_1}(s_1)) = \mathrm{G}_{\mathbf{p}}^{\mathbf{p}_1}(\tau_{\mathbf{p}_2}(s_2))$, whence $\tau_{\mathbf{p}}\left(\mathrm{F}_{\mathbf{p}}^{\mathbf{p}_1}(s_1)\right) = \tau_{\mathbf{p}}\left(\mathrm{F}_{\mathbf{p}}^{\mathbf{p}_2}(s_2)\right)$; since $\tau_{\mathbf{p}}$ is injective, $\mathrm{F}_{\mathbf{p}}^{\mathbf{p}_1}(s_1) = \mathrm{F}_{\mathbf{p}}^{\mathbf{p}_2}(s_2)$, whence $x_1 = x_2$. ✠

Proposition A.4.7. *Let* $\mathbf{P}$ *be a directed preorder,* τ *an algebraic type,* $\mathrm{E}, \mathrm{F}, \mathrm{G}$ *functors from* $\mathbf{P}$ *to* $\mathbf{Alg}_\tau$ *with direct limits* $(\mathbf{e}, \epsilon)$*,* $(\mathbf{f}, \phi)$*,* $(\mathbf{g}, \gamma)$*;* α, β *natural transformations from* F *to* G*,* δ *a natural transformation from* E *to* F *such that, for each* $\mathbf{p} \in \mathcal{O}(\mathbf{P})$*,* $\delta_{\mathbf{p}}$ *is an inverse equalizer for* $\alpha_{\mathbf{p}}$ *and* $\beta_{\mathbf{p}}$*. Let* $\delta^* = \varinjlim[\delta, (e, \epsilon), (f, \phi)]$*,* $\alpha^* = \varinjlim[\alpha, (f, \phi), (g, \gamma)]$*,* $\beta^* = \varinjlim[\beta, (f, \phi), (g, \gamma)]$*. Then* $\mathrm{Img}(\delta^*) = \mathbf{s}(\alpha^*, \beta^*)$*. In particular,* δ^* *is an inverse equalizer for* α^*, β^**.*

Proof. Let $t \in \delta^*(\mathbf{e})$. Then $t = \delta^*(\epsilon_{\mathbf{p}}(s))$ for some $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and some $s \in \mathrm{E}(\mathbf{p})$, that is $t = \phi_{\mathbf{p}}(\delta_{\mathbf{p}}(s))$, so

$$\alpha^*(t) = \alpha^*(\phi_{\mathbf{p}}(\delta_{\mathbf{p}}(s))) = \gamma_{\mathbf{p}}(\alpha_{\mathbf{p}}(\delta_{\mathbf{p}}(s))) = \gamma_{\mathbf{p}}(\beta_{\mathbf{p}}(\delta_{\mathbf{p}}(s))) = \beta^*(\phi_{\mathbf{p}}(\delta_{\mathbf{p}}(s))) = \beta^*(t)$$

that is, $t \in \mathbf{s}$.

Let $t \in \mathbf{s}$. Then $t = \phi_{\mathbf{p}}(s)$ for some $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and some $s \in \mathrm{F}(\mathbf{p})$, and $\alpha^*(\phi_{\mathbf{p}}(s)) = \beta^*(\phi_{\mathbf{p}}(s))$, whence $\gamma_{\mathbf{p}}(\alpha_{\mathbf{p}}(s)) = \gamma_{\mathbf{p}}(\beta_{\mathbf{p}}(s))$; thus there is $\mathbf{q} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p} \preccurlyeq \mathbf{q}$ and $\mathrm{G}^{\mathbf{p}}_{\mathbf{q}}(\alpha_{\mathbf{p}}(s)) = \mathrm{G}^{\mathbf{p}}_{\mathbf{q}}(\beta_{\mathbf{p}}(s))$, that is $\alpha_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s)) = \beta_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s))$; by Proposition A.2.37 there is $u \in \mathrm{E}(\mathbf{q})$ such that $\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s) = \delta_{\mathbf{q}}(u)$, thus $t = \phi_{\mathbf{p}}(s) = \phi_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s)) = \phi_{\mathbf{q}}(\delta_{\mathbf{q}}(u)) = \delta^*(\epsilon_{\mathbf{q}}(u))$, that is, $t \in \delta^*(\mathbf{e})$. ✠

Proposition A.4.8. *Let* $\mathbf{P}$ *be a directed preorder,* $\mathbf{C}$ *a concrete category,* F *and* G *functors from* $\mathbf{P}$ *to* $\mathbf{C}$*,* τ *a natural transformation from* F *to* G *such that, for each* $\mathbf{p} \in \mathcal{O}(\mathbf{P})$*,* $\tau_{\mathbf{p}}$ *is surjective. Then any direct limit of* τ *is surjective.*

Proof. Let $\bar{\tau}$ be the direct limit of τ relative to the direct limits $(\mathbf{f}, \phi)$ of F and $(\mathbf{g}, \gamma)$ of G. If $y \in \mathbf{g}$ there are $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and $s \in \mathrm{G}(\mathbf{p})$ such that $y = \gamma_{\mathbf{p}}(s)$. Since $\tau_{\mathbf{p}}$ is surjective, there is $t \in \mathrm{F}(\mathbf{p})$ such that $s = \tau_{\mathbf{p}}(t)$; if $x = \phi_{\mathbf{p}}(t)$, then $\bar{\tau}(x) = \bar{\tau}(\phi_{\mathbf{p}}(t)) = \gamma_{\mathbf{p}}(\tau_{\mathbf{p}}(t)) = \gamma_{\mathbf{p}}(s) = y$. ✠

Proposition A.4.9. *Let* $\mathbf{P}$ *be a directed preorder,* τ *an algebraic type,* E, F, G *functors from* $\mathbf{P}$ *to* $\mathbf{Alg}_\tau$ *with direct limits* $(\mathbf{e}, \epsilon)$*,* $(\mathbf{f}, \phi)$*,* $(\mathbf{g}, \gamma)$*;* α, β *natural transformations from* E *to* F*,* δ *a natural transformation from* F *to* G *such that, for each* $\mathbf{p} \in \mathcal{O}(\mathbf{P})$*,* $\delta_{\mathbf{p}}$ *is a direct equalizer for* $\alpha_{\mathbf{p}}$ *and* $\beta_{\mathbf{p}}$*. Let* $\delta^* = \varinjlim[\delta, (e, \epsilon), (f, \phi)]$*,* $\alpha^* = \varinjlim[\alpha, (f, \phi), (g, \gamma)]$*,* $\beta^* = \varinjlim[\beta, (f, \phi), (g, \gamma)]$*. Then* $\mathrm{Coi}(\delta^*) = \mathbf{g}/\mathbf{r}(\alpha^*, \beta^*)$*. In particular,* δ^* *is a direct equalizer for* α^*, β^**.*

Proof. Let $(x, y) \in \mathbf{eq}(\delta^*)$. Since $x = \phi_{\mathbf{p}}(s)$ and $y = \phi_{\mathbf{p}}(f)$ for some $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and some $s \in \mathrm{G}(\mathbf{p})$, $t \in \mathrm{G}(\mathbf{p})$, we have $\delta^*(\phi_{\mathbf{p}}(s)) = \delta^*(\phi_{\mathbf{p}}(t))$, whence $\gamma_{\mathbf{p}}(\delta_{\mathbf{p}}(s)) = \gamma_{\mathbf{p}}(\delta_{\mathbf{p}}(t))$; then there is $\mathbf{q} \in \mathcal{O}(\mathbf{P})$ such that $\mathbf{p} \preccurlyeq \mathbf{q}$ and $\mathrm{G}^{\mathbf{p}}_{\mathbf{q}}(\delta_{\mathbf{p}}(s)) = \mathrm{G}^{\mathbf{p}}_{\mathbf{q}}(\delta_{\mathbf{p}}(t))$, whence $\delta_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s)) = \delta_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(t))$, that is, $\left(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s), \mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(t)\right) \in \mathbf{eq}(\delta_{\mathbf{q}})$; by Proposition A.2.27 $\mathbf{eq}(\delta_{\mathbf{q}}) = \mathbf{r}(\alpha_{\mathbf{q}}, \beta_{\mathbf{q}})$; now let $\mathbf{l}$ be a congruence on $\mathbf{g}$ such that $r_0(\alpha^*, \beta^*) \subseteq \mathbf{l}$; then $\mathbf{l} = \mathbf{eq}(h)$, where

$$\begin{aligned} h : \mathbf{g} &\to \mathbf{g}/\mathbf{l} \\ x &\mapsto [x]_{\mathbf{l}}; \end{aligned}$$

also, for each $\mathbf{p} \in \mathcal{O}(\mathbf{P})$, $h \circ \phi_{\mathbf{p}} \circ \alpha_{\mathbf{p}} = h \circ \alpha^* \circ \epsilon_{\mathbf{p}} = h \circ \beta^* \circ \epsilon_{\mathbf{p}} = h \circ \phi_{\mathbf{p}} \circ \beta_{\mathbf{p}}$, which yields $r_0(\alpha_{\mathbf{p}}, \beta_{\mathbf{p}}) \subseteq \mathbf{eq}(h \circ \phi_{\mathbf{p}})$. In particular $r_0(\alpha_{\mathbf{q}}, \beta_{\mathbf{q}}) \subseteq \mathbf{eq}(h \circ \phi_{\mathbf{q}})$, so eventually

$$h(x) = h(\phi_{\mathbf{p}}(s)) = h(\phi_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(s))) = h(\phi_{\mathbf{q}}(\mathrm{F}^{\mathbf{p}}_{\mathbf{q}}(t))) = h(\phi_{\mathbf{p}}(t)) = h(y)$$

that is, $(x, y) \in \mathbf{l}$ and $\mathbf{eq}(\delta^*) \subseteq \mathbf{l}$. Since $\mathbf{l}$ is any congruence on $\mathbf{g}$ such that $r_0(\alpha^*, \beta^*) \subseteq \mathbf{l}$, we have $\mathbf{r}(\alpha^*, \beta^*) \subseteq \mathbf{eq}(\delta^*)$

Let $(x, y) \in r_0(\alpha^*, \beta^*)$. Then $x = \alpha^*(z)$, $y = \beta^*(z)$ for a $z \in \mathbf{e}$, and $z = \epsilon_{\mathbf{p}}(s)$ for some $\mathbf{p} \in \mathcal{O}(\mathbf{P})$ and $s \in \mathrm{E}(\mathbf{p})$. Thus

$$\begin{aligned} \delta^*(x) &= \delta^*(\alpha^*(z)) = \delta^*(\alpha^*(\epsilon_{\mathbf{p}}(s))) = \delta^*(\phi_{\mathbf{p}}(\alpha_{\mathbf{p}}(s))) = \gamma_{\mathbf{p}}(\delta_{\mathbf{p}}(\alpha_{\mathbf{p}}(s))) = \\ &= \gamma_{\mathbf{p}}(\delta_{\mathbf{p}}(\beta_{\mathbf{p}}(s))) = \delta^*(\phi_{\mathbf{p}}(\beta_{\mathbf{p}}(s))) = \delta^*(\beta^*(\epsilon_{\mathbf{p}}(s))) = \delta^*(\beta^*(z)) = \delta^*(y) \end{aligned}$$

that is, $(x, y) \in \mathbf{eq}(\delta^*)$ and $\mathbf{r}(\alpha^*, \beta^*) \subseteq \mathbf{eq}(\delta^*)$. ✠

A.5 Adjunction

Definition A.5.1. Let $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ and $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ be functors. An ***adjunction from*** F **to** G is a natural isomorphism $\phi : \overline{\mathbf{D}} \circ (\mathrm{F} \times \mathrm{I}_{\mathbf{D}}) \to \overline{\mathbf{C}} \circ (\mathrm{I}_{\mathbf{C}} \times \mathrm{G})$. That is

$$\begin{aligned} \phi :\ & \mathcal{O}(\mathbf{C}) \times \mathcal{O}(\mathbf{D}) \to \mathcal{M}(\mathbf{Set}) \\ & (\mathbf{c}, \mathbf{d}) \mapsto \phi_{\mathbf{c},\mathbf{d}} \end{aligned}$$

where, for each pair $(\mathbf{c}, \mathbf{d}) \in \mathcal{O}(\mathbf{C}) \times \mathcal{O}(\mathbf{D})$, $\phi_{\mathbf{c},\mathbf{d}}$ is a bijection

$$\phi_{\mathbf{c},\mathbf{d}} : \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}) \to \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d}))$$

and for each $f \in \overline{\mathbf{C}}\,(\mathbf{c}, \mathbf{c}_*)$ and $g \in \overline{\mathbf{D}}\,(\mathbf{d}, \mathbf{d}_*)$ the diagram

$$\begin{array}{ccc} \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}_*), \mathbf{d}) & \xrightarrow{\phi_{\mathbf{c}_*,\mathbf{d}}} & \overline{\mathbf{C}}\,(\mathbf{c}_*, \mathrm{G}(\mathbf{d})) \\ {\scriptstyle \overline{\mathbf{D}}(\mathrm{F}(f),g)}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(f,\mathrm{G}(g))} \\ \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}_*) & \xrightarrow{\phi_{\mathbf{c},\mathbf{d}_*}} & \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d}_*)) \end{array}$$

commutes, or both the diagrams

$$\begin{array}{ccc} \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}_*), \mathbf{d}) & \xrightarrow{\phi_{\mathbf{c}_*,\mathbf{d}}} & \overline{\mathbf{C}}\,(\mathbf{c}_*, \mathrm{G}(\mathbf{d})) \\ {\scriptstyle \overline{\mathbf{D}}(\mathrm{F}(f),\mathbf{d})}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(f,\mathrm{G}(\mathbf{d}))} \\ \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}) & \xrightarrow{\phi_{\mathbf{c},\mathbf{d}}} & \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d})) \end{array}$$

$$\begin{array}{ccc} \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}) & \xrightarrow{\phi_{\mathbf{c},\mathbf{d}}} & \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d})) \\ {\scriptstyle \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}),g)}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(\mathbf{c},\mathrm{G}(g))} \\ \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}_*) & \xrightarrow{\phi_{\mathbf{c},\mathbf{d}_*}} & \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d}_*)) \end{array}$$

commute.
More in details, for each $h \in \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d})$ and $k \in \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}_*), \mathbf{d})$

$$\begin{aligned} &\phi_{\mathbf{c},\mathbf{d}_*}\,(g \circ h) = \mathrm{G}(g) \circ \phi_{\mathbf{c},\mathbf{d}}(h) \\ &\phi_{\mathbf{c},\mathbf{d}}\,(k \circ \mathrm{F}(f)) = \phi_{\mathbf{c}_*,\mathbf{d}}(k) \circ f \end{aligned}$$

and for each $k \in \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d}))$ and $l \in \overline{\mathbf{C}}\,(\mathbf{c}_*, \mathrm{G}(\mathbf{d}))$

$$\begin{aligned} &\phi^{-1}_{\mathbf{c},\mathbf{d}_*}\,(\mathrm{G}(g) \circ k) = g \circ \phi^{-1}_{\mathbf{c},\mathbf{d}}(k) \\ &\phi^{-1}_{\mathbf{c},\mathbf{d}}\,(l \circ f) = \phi^{-1}_{\mathbf{c}_*,\mathbf{d}}(k) \circ \mathrm{F}(f). \end{aligned}$$

Proposition A.5.1. *Let* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ *and* $\mathrm{G}{:}\mathbf{C}{\to}\mathbf{D}$ *be functors,* ϕ *an adjunction from* F *to* G*,* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$*. Set* $\eta_{\mathbf{c}} = \phi_{\mathbf{c},\mathrm{F}(\mathbf{c})}(\mathrm{id}_{\mathrm{F}(\mathbf{c})})$*. Then* $(\mathrm{F}(\mathbf{c}), \eta_{\mathbf{c}})$ *is a universal arrow from* $\mathbf{c}$ *to* G*, for* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$ *and* $f \in \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}), \mathbf{d})$ $\phi_{\mathbf{c},\mathbf{d}}(f) = \mathrm{G}(f) \circ \eta_{\mathbf{c}}$*, and the map*

$$\begin{aligned} \eta :\ & \mathcal{O}(\mathbf{C}) \to \mathcal{M}(\mathbf{C}) \\ & \mathbf{c} \mapsto \eta_{\mathbf{c}} \end{aligned}$$

is a natural transformation from $\mathrm{I}_{\mathbf{C}}$ *to* $\mathrm{G} \circ \mathrm{F}$*.*
Conversely, if η *is a natural transformation from* $\mathrm{I}_{\mathbf{C}}$ *to* $\mathrm{G} \circ \mathrm{F}$ *and for each* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ $(\mathrm{F}(\mathbf{c}), \eta_{\mathbf{c}})$ *is a universal arrow from* $\mathbf{c}$ *to* G*, then there is an adjunction* ϕ *from* F *to* G *such that*

$$\begin{aligned} \phi_{\mathbf{c},\mathbf{d}} :\ & \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}), \mathbf{d}) \to \overline{\mathbf{C}}(\mathbf{c}, \mathrm{G}(\mathbf{d})) \\ & f \mapsto \mathrm{G}(f) \circ \eta_{\mathbf{c}} \end{aligned}$$

for each $(\mathbf{c}, \mathbf{d}) \in \mathcal{O}(\mathbf{C}) \times \mathcal{O}(\mathbf{D})$*.*

Proof. Suppose that ϕ is an adjunction from F to G. For $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ set

$$\begin{aligned} \phi^{(\mathbf{c})}_{\mathbf{d}} :\ & \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}), \mathbf{d}) \to \overline{\mathbf{C}}(\mathbf{c}, \mathrm{G}(\mathbf{d})) \\ & h \mapsto \phi_{\mathbf{c},\mathbf{d}}(h). \end{aligned}$$

Then $\phi^{(\mathbf{c})}_{\mathbf{d}} : \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}), -) \to \overline{\mathbf{C}}(\mathbf{c}, \mathrm{G}-)$ is a natural isomorphism, so by PropositionA.2.11 $\big(\mathrm{F}(\mathbf{c}), \phi^{(\mathbf{c})}_{\mathrm{F}(\mathbf{c})}(\mathrm{id}_{\mathrm{F}(\mathbf{c})})\big)$ is a universal arrow from $\mathbf{c}$ to G.
For any $\mathbf{c}_1 \in \mathcal{O}(\mathbf{C})$, $\mathbf{c}_2 \in \mathcal{O}(\mathbf{C})$ and $f : \mathbf{c}_1 \to \mathbf{c}_2$ the diagram

$$\begin{array}{ccc} \mathrm{I}_{\mathbf{C}}(\mathbf{c}_1) & \xrightarrow{\eta_{\mathbf{c}_1}} & \mathrm{G} \circ \mathrm{F}(\mathbf{c}_1) \\ {\scriptstyle \mathrm{I}_{\mathbf{C}}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{G}\circ\mathrm{F}(f)} \\ \mathrm{I}_{\mathbf{C}}(\mathbf{c}_2) & \xrightarrow{\eta_{\mathbf{c}_2}} & \mathrm{G} \circ \mathrm{F}(\mathbf{c}_2) \end{array}$$

commutes because

$$\begin{aligned} \eta_{\mathbf{c}_2} \circ \mathrm{I}_{\mathbf{C}}(f) = \eta_{\mathbf{c}_2} \circ f = \phi^{(\mathbf{c}_2)}_{\mathrm{F}(\mathbf{c}_2)}(\mathrm{id}_{\mathrm{F}(\mathbf{c}_2)}) \circ f = \phi^{(\mathbf{c}_1)}_{\mathrm{F}(\mathbf{c}_2)}(\mathrm{id}_{\mathrm{F}(\mathbf{c}_2)} \circ \mathrm{F}(f)) = \\ = \phi^{(\mathbf{c}_1)}_{\mathrm{F}(\mathbf{c}_2)}(\mathrm{F}(f) \circ \mathrm{id}_{\mathrm{F}(\mathbf{c}_1)}) = \mathrm{G}(\mathrm{F}(f)) \circ \phi^{(\mathbf{c}_1)}_{\mathrm{F}(\mathbf{c}_1)}(\mathrm{id}_{\mathrm{F}(\mathbf{c}_1)}) = \\ = (\mathrm{G} \circ \mathrm{F})(f) \circ \eta_{\mathbf{c}_1}. \end{aligned}$$

If for each $c \in \mathcal{O}(\mathbf{C})$ $(\mathrm{F}(\mathbf{c}), \eta_{\mathbf{c}})$ is a universal arrow from $\mathbf{c}$ to G, then by Proposition A.2.9

$$\begin{aligned} \phi^{(\mathbf{c})}_{\mathbf{d}} :\ & \overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}), \mathbf{d}) \to \overline{\mathbf{C}}(\mathbf{c}, \mathrm{G}(\mathbf{d})) \\ & f \mapsto \mathrm{G}(f) \circ \eta_{\mathbf{c}} \end{aligned}$$

define a natural isomorphism $\phi^{(\mathbf{c})} : \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), -) \to \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}-)$. If the map η is a natural transformation $\eta :\ \mathrm{I}_{\mathbf{C}} \to \mathrm{G} \circ \mathrm{F}$, then if $f : \mathbf{c} \to \mathbf{c}_*$ the diagram

$$\begin{array}{ccc}
\overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}_*), \mathbf{d}) & \xrightarrow{\phi_{\mathbf{d}}^{(\mathbf{c}*)}} & \overline{\mathbf{C}}\,(\mathbf{c}_*, \mathrm{G}(\mathbf{d})) \\
{\scriptstyle \overline{\mathbf{D}}(\mathrm{F}(f),\mathbf{d})}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(f,\mathrm{G}(\mathbf{d}))} \\
\overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}) & \xrightarrow{\phi_{\mathbf{d}}^{(\mathbf{c})}} & \overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d}))
\end{array}$$

commutes because, for $h : \mathrm{F}(\mathbf{c}) \to \mathbf{d}$

$$\begin{aligned}
\phi_{\mathbf{d}}^{(\mathbf{c})}\left(\overline{\mathbf{D}}\,(\mathrm{F}(f), \mathbf{d})\,(h)\right) &= \mathrm{G}\,(h \circ \mathrm{F}(f)) \circ \eta_{\mathbf{c}} = \mathrm{G}(h) \circ \mathrm{G}(\mathrm{F}(f)) \circ \eta_{\mathbf{c}} = \\
&= \mathrm{G}(h) \circ (\mathrm{G} \circ \mathrm{F})\,(f) \circ \eta_{\mathbf{c}} = \mathrm{G}(h) \circ \eta_{\mathbf{c}_*} \circ f = \\
&= \overline{\mathbf{C}}\,(f, \mathrm{G}(\mathbf{d}))\,(\phi_{\mathbf{d}}^{(\mathbf{c}*)}(h))\,.
\end{aligned}$$

So

$$\begin{aligned}
\phi :\ &\mathcal{O}(\mathbf{C}) \times \mathcal{O}(\mathbf{D}) \to \mathcal{M}(\mathbf{Set}) \\
&(\mathbf{c}, \mathbf{d}) \mapsto \phi_{\mathbf{c},\mathbf{d}}
\end{aligned}$$

is an adjunction from F to G. ✠

Definition A.5.2. Let ϕ be an adjunction from F to G. The natural transformation $\eta : \mathrm{I}_{\mathrm{dom}\,\mathrm{F}} \to \mathrm{GF}$ defined for $\mathbf{c} \in \mathcal{O}(\mathrm{dom}\,\mathrm{F})$ by $\eta_{\mathbf{c}} = \phi_{\mathbf{c},\mathrm{F}(\mathbf{c})}(\mathrm{id}_{\mathrm{F}(\mathbf{c})})$ is called the ***unit*** of ϕ.

Proposition A.5.2. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$ *and* $\mathrm{G}:\mathbf{D}\to\mathbf{C}$ *be functors,* ϕ *an adjunction from* F *to* G*,* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$*. Set* $\varepsilon_{\mathbf{d}} = \phi^{-1}_{\mathrm{G}(\mathbf{d}),\mathbf{d}}(\mathrm{id}_{\mathrm{G}(\mathbf{D})})$*. Then* $(\mathrm{G}(\mathbf{d}), \varepsilon_{\mathbf{d}})$ *is a universal arrow from* F *to* $\mathbf{d}$*, for* $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ *and* $g \in \overline{\mathbf{C}}\,(\mathbf{d}, \mathrm{G}(\mathbf{c}))$ $\phi^{-1}_{\mathbf{c},\mathbf{d}}(f) = \varepsilon_{\mathbf{d}} \circ \mathrm{F}(g)$*, and the map*

$$\begin{aligned}
\varepsilon :\ &\mathcal{O}(\mathbf{D}) \to \mathcal{M}(\mathbf{D}) \\
&\mathbf{d} \mapsto \varepsilon_{\mathbf{d}}
\end{aligned}$$

is a natural transformation from $\mathrm{F} \circ \mathrm{G}$ *to* $\mathrm{I}_{\mathbf{D}}$*.*
Conversely, if ε *is a natural transformation from* $\mathrm{F} \circ \mathrm{G}$ *to* $\mathrm{I}_{\mathbf{D}}$ *and for each* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$ $(\mathrm{G}(\mathbf{d}), \varepsilon_{\mathbf{d}})$ *is a universal arrow from* F *to* $\mathbf{d}$*, then there is an adjunction* ϕ *from* F *to* G *such that*

$$\begin{aligned}
\phi^{-1}_{\mathbf{c},\mathbf{d}} :\ &\overline{\mathbf{C}}\,(\mathbf{c}, \mathrm{G}(\mathbf{d})) \to \overline{\mathbf{D}}\,(\mathrm{F}(\mathbf{c}), \mathbf{d}) \\
&g \mapsto \varepsilon_{\mathbf{d}} \circ \mathrm{F}(g)
\end{aligned}$$

for each $(\mathbf{c}, \mathbf{d}) \in \mathcal{O}(\mathbf{C}) \times \mathcal{O}(\mathbf{D})$*.*

Proof. Analogous to the proof of Proposition A.5.1. ✠

Definition A.5.3. Let ϕ be an adjunction from F to G. The natural transformation $\varepsilon : \mathrm{FG} \to \mathrm{I}_{\mathrm{dom}\,\mathrm{G}}$ defined for $\mathbf{d} \in \mathcal{O}(\mathrm{dom}\,\mathrm{G})$ by $\varepsilon_{\mathbf{d}} = \phi^{-1}_{\mathrm{G}(\mathbf{d}),\mathbf{d}}(\mathrm{id}_{\mathrm{G}(\mathbf{d})})$ is called the ***counit*** of ϕ.

Proposition A.5.3. *Let* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$*,* $\mathrm{G}{:}\mathbf{D}{\to}\mathbf{C}$ *be functors and* ϕ *be an adjunction from* F *to* G*,* η *and* ε *its unit and counit. Then for every* $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ *and* $\mathbf{d}\in\mathcal{O}(\mathbf{D})$

$$\mathrm{G}(\varepsilon_{\mathbf{d}})\eta_{\mathrm{G}(\mathbf{d})} = \mathrm{id}_{\mathrm{G}(\mathbf{d})} \tag{A.120}$$

$$\varepsilon_{\mathrm{F}(\mathbf{c})}\mathrm{F}(\eta_{\mathbf{c}}) = \mathrm{id}_{\mathrm{F}(\mathbf{c})}\,. \tag{A.121}$$

Proof. By the definitions of unit and counit of an adjunction and Propositions A.5.1 and A.5.2

$$\mathrm{id}_{\mathrm{G}(\mathbf{d})} = \phi_{\mathrm{G}(\mathbf{d}),\mathbf{d}}(\varepsilon_{\mathbf{d}}) = \mathrm{G}(\varepsilon_{\mathbf{d}})\eta_{\mathrm{G}(\mathbf{d})} \tag{A.122}$$

and

$$\mathrm{id}_{\mathrm{F}(\mathbf{c})} = \phi^{-1}_{\mathbf{c},\mathrm{F}(\mathbf{c})}(\eta_{\mathbf{c}}) = \varepsilon_{\mathrm{F}(\mathbf{c})}\mathrm{F}(\eta_{\mathbf{c}}). \tag{A.123}$$

✠

Proposition A.5.4. *Let* $\mathrm{G}{:}\mathbf{D}{\to}\mathbf{C}$ *be a functor, and for each* $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ *let* $(\mathrm{F}_{\mathbf{c}},\eta_{\mathbf{c}})$ *be a universal arrow from* $\mathbf{c}$ *to* G*. Then there is a functor* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ *such that for* $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ $\mathrm{F}(\mathbf{c}) = \mathrm{F}_{\mathbf{c}}$*, and the* $\eta_{\mathbf{c}}$ *are the components of a natural transformation* $\eta : \mathrm{I}_{\mathbf{C}} \to \mathrm{GF}$*. Thus* F *is a left adjoint to* G*.*

Proof. If $f : \mathbf{c}_1 \to \mathbf{c}_2$ define $\mathrm{F}(f)$ as the unique map such that $\eta_{\mathbf{c}_2}f = \mathrm{GF}(f)\eta_{\mathbf{c}_1}$. It's easy to check that this defines a functor $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$. This definition also makes $\eta_{\mathbf{c}}$ into the components of a natural transformation $\eta : \mathrm{I}_{\mathbf{C}} \to \mathrm{GF}$. ✠

Proposition A.5.5. *Let* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$ *be a functor, and for each* $\mathbf{d}\in\mathcal{O}(\mathbf{D})$ *let* $(\mathrm{G}_{\mathbf{d}},\varepsilon_{\mathbf{d}})$ *be a universal arrow from* F *to* $\mathbf{d}$*. Then there is a functor* $\mathrm{G}{:}\mathbf{D}{\to}\mathbf{C}$ *such that for* $\mathbf{d}\in\mathcal{O}(\mathbf{D})$ $\mathrm{G}(\mathbf{d}) = \mathrm{G}_{\mathbf{D}}$*, and the* $\varepsilon_{\mathbf{d}}$ *are the components of a natural transformation* $\varepsilon : \mathrm{FG} \to \mathrm{I}_{\mathbf{D}}$*. Thus* G *is a right adjoint to* F*.*

Proof. If $f : \mathbf{d}_1 \to \mathbf{d}_2$ define $\mathrm{G}(f)$ as the unique map such that $\varepsilon_{\mathbf{d}_2}\mathrm{FG}(f) = f\varepsilon_{\mathbf{d}_1}$. It's easy to check that this defines a functor $\mathrm{G}{:}\mathbf{D}{\to}\mathbf{C}$. This definition also makes $\varepsilon_{\mathbf{d}}$ into the components of a natural transformation $\varepsilon : \mathrm{FG} \to \mathrm{I}_{\mathbf{D}}$. ✠

Proposition A.5.6. *Let* $\mathrm{F}{:}\mathbf{C}{\to}\mathbf{D}$*,* $\mathrm{G}{:}\mathbf{C}{\to}\mathbf{D}$*,* $\mathrm{H}{:}\mathbf{D}{\to}\mathbf{E}$*,* $\mathrm{K}{:}\mathbf{B}{\to}\mathbf{C}$ *be functors and* $\tau : \mathrm{F} \to \mathrm{G}$*. Then there are natural transformation* $\mathrm{H}\tau : \mathrm{HF} \to \mathrm{HG}$ *and* $\tau\mathrm{K} : \mathrm{FK} \to \mathrm{GK}$ *defined for* $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ *and* $\mathbf{b}\in\mathcal{O}(\mathbf{B})$ *by*

$$\mathrm{H}\tau_{\mathbf{c}} = \mathrm{H}(\tau_{\mathbf{c}}) \tag{A.124}$$

end

$$\tau\mathrm{F}_{\mathbf{b}} = \tau_{\mathrm{K}(\mathbf{b})}. \tag{A.125}$$

Proof. Routine check. ✠

Proposition A.5.7. *Let* $\mathrm{F}:\mathbf{C}\to\mathbf{D}$, $\mathrm{G}:\mathbf{D}\to\mathbf{C}$ *be functors and* $\eta : \mathrm{I}_{\mathbf{C}} \to \mathrm{GF}$, $\varepsilon : \mathrm{FG} \to \mathrm{I}_{\mathbf{D}}$ *natural transformation such that*

$$\mathrm{G}\varepsilon \circ \eta\mathrm{G} = \mathrm{id}_{\mathrm{G}} \tag{A.126}$$

$$\varepsilon\mathrm{F} \circ \mathrm{F}\eta = \mathrm{id}_{\mathrm{F}}\,. \tag{A.127}$$

Then there is an adjunction from F *to* G *of which* η *is the unit and* ε *is the counit.*

Proof. Let $\mathbf{c} \in \mathcal{O}(\mathbf{C})$, we prove that $(\mathrm{F}(\mathbf{c}), \eta_{\mathbf{c}})$ is a universal arrow from $\mathbf{c}$ to G.
Let $f : \mathbf{c} \to \mathrm{G}(\mathbf{d})$, and set $h = \varepsilon_{\mathbf{d}} \circ \mathrm{F}(f)$. We prove that $\mathrm{G}(h) \circ \eta_{\mathbf{c}} = f$. Since $\eta : \mathrm{I}_{\mathbf{C}} \to \mathrm{GF}$ is a natural transformation

$$\mathrm{G}\varepsilon_{\mathbf{d}} \circ \mathrm{GF}(f) \circ \eta_{\mathbf{c}} = \mathrm{G}\varepsilon_{\mathbf{d}} \circ \eta\mathrm{G}_{\mathbf{d}} \circ f = \mathrm{id}_{\mathrm{G}(\mathbf{d})} \circ f = f. \tag{A.128}$$

If $g : \mathrm{F}(\mathbf{c}) \to \mathbf{d}$ is such that $\mathrm{G}(g) \circ \eta_{\mathbf{c}} = f$ then

$$h = \varepsilon_{\mathbf{d}} \circ \mathrm{F}(f) = \varepsilon_{\mathbf{d}} \circ \mathrm{FG}(g) \circ \mathrm{F}(\eta_{\mathbf{c}}) = g \circ \varepsilon\mathrm{F}_{\mathbf{c}} \circ \mathrm{F}\eta_{\mathbf{c}} = g \circ \mathrm{id}_{\mathrm{F}(\mathbf{c})} = g. \tag{A.129}$$

✠

Proposition A.5.8. *A right adjoint functor preserves inverse limits.*

Proof. Let $\mathrm{F}:\mathbf{C}\to\mathbf{D}$, $\mathrm{g}:\mathbf{D}\to\mathbf{C}$, and φ an adjunction from F to G. Let $\mathrm{H}:\mathbf{J}\to\mathbf{D}$ and $(\mathbf{l}, \lambda)$ be an inverse limit for H. Then $(\mathrm{G}(\mathbf{l}), \mathrm{G}\lambda)$ is an inverse target for GH. Let $(\mathbf{t}, \tau)$ be another inverse target for GH. Then for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ $\tau_{\mathbf{j}} : \mathbf{t} \to \mathrm{GH}(\mathbf{j})$. For $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ set $\mu_{\mathbf{j}} = \varphi^{-1}_{\mathbf{c},\mathrm{H}(\mathbf{j})}(\tau_{\mathbf{j}}) : \mathrm{F}(\mathbf{t}) \to \mathrm{H}(j)$. If $f \in \overline{\mathbf{J}}(\mathbf{i}, \mathbf{j})$ the diagram

$$\begin{array}{ccc} \overline{\mathbf{D}}(\mathrm{F}(\mathbf{t}), \mathrm{H}(\mathbf{i})) & \xrightarrow{\varphi_{\mathbf{t},\mathrm{H}(\mathbf{i})}} & \overline{\mathbf{C}}(\mathbf{t}, \mathrm{GH}(\mathbf{i})) \\ {\scriptstyle \overline{\mathbf{D}}(\mathrm{F}(\mathbf{t}),\mathrm{H}(f))}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(\mathbf{t},\mathrm{GH}(f))} \\ \overline{\mathbf{D}}(\mathrm{F}(\mathbf{t}), \mathrm{H}(\mathbf{j})) & \xrightarrow{\varphi_{\mathbf{t},\mathrm{H}(\mathbf{j})}} & \overline{\mathbf{C}}(\mathbf{t}, \mathrm{GH}(\mathbf{j})) \end{array} \tag{A.130}$$

commutes, thus

$$\mathrm{H}(f) \circ \mu_{\mathbf{i}} = \mathrm{H}(f) \circ \varphi^{-1}_{\mathbf{t},\mathrm{H}(\mathbf{i})}(\tau_{\mathbf{i}}) = \varphi^{-1}_{\mathbf{t},\mathrm{H}(\mathbf{j})}(\mathrm{GH}(f) \circ \tau_i) = \varphi^{-1}_{\mathbf{t},\mathrm{H}(\mathbf{j})}(\tau_j) = \mu_{\mathbf{j}} \tag{A.131}$$

and so the $\tau_{\mathbf{i}}$ are the components of a natural transformation $\tau : \mathrm{C}^{\mathbf{J},\mathbf{D}}_{\mathrm{F}(\mathbf{t})} \to \mathrm{H}$, that is, $(\mathrm{F}(\mathbf{t}), \mu)$ is an inverse target for H. Thus there exists a unique $h : \mathrm{F}(\mathbf{t}) \to \mathbf{l}$ such that $\mu = \lambda \circ \gamma_h^{\mathbf{J},\mathbf{D}}$. Since for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ the diagram

$$\begin{array}{ccc} \overline{\mathbf{D}}(\mathrm{F}(\mathbf{t}), \mathbf{l}) & \xrightarrow{\varphi_{\mathbf{t},\mathbf{l}}} & \overline{\mathbf{C}}(\mathbf{t}, \mathrm{G}(\mathbf{l})) \\ {\scriptstyle \overline{\mathbf{D}}(\mathrm{F}(\mathbf{t}),\lambda_{\mathbf{i}})}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{C}}(\mathbf{t},\mathrm{G}(\lambda_{\mathbf{j}}))} \\ \overline{\mathbf{D}}(\mathrm{F}(\mathbf{t}), \mathrm{H}(\mathbf{j})) & \xrightarrow{\varphi_{\mathbf{t},\mathrm{H}(\mathbf{j})}} & \overline{\mathbf{C}}(\mathbf{t}, \mathrm{GH}(\mathbf{j})) \end{array} \tag{A.132}$$

commutes, we have

$$G\lambda_{\mathbf{j}} \circ \varphi_{\mathbf{t},\mathbf{l}}(h) = \varphi_{\mathbf{t},\mathrm{H}(\mathbf{j})}(\lambda_{\mathbf{i}} \circ h) = \varphi_{\mathbf{t},\mathrm{H}(\mathbf{j})}(\mu_{\mathbf{j}}) = \tau_{\mathbf{j}}. \tag{A.133}$$

and thus

$$G\lambda \circ \gamma^{\mathbf{J},\mathbf{D}}_{\varphi_{\mathbf{t},\mathbf{l}}(h)} = \tau. \tag{A.134}$$

If also $k : \mathbf{t} \to \mathrm{G}(\mathbf{l})$ is such that

$$G\lambda \circ \gamma^{\mathbf{J},\mathbf{D}}_{k} = \tau \tag{A.135}$$

then again because the diagram A.132 commutes we have for $\mathbf{j} \in \mathcal{O}(\mathbf{J})$

$$\lambda_j \varphi^{-1}_{\mathbf{t},\mathbf{l}}(k) = \varphi^{-1}_{\mathbf{t},\mathrm{H}(\mathbf{j})}(\mathrm{G}(\lambda_{\mathbf{j}}) \circ k) = \varphi^{-1}_{\mathbf{t},\mathrm{H}(\mathbf{j})}(\tau_{\mathbf{j}}) = \mu_{\mathbf{j}} \tag{A.136}$$

whence $\varphi^{-1}_{\mathbf{t},\mathbf{l}}(k) = h$ and $k = \varphi_{\mathbf{t},\mathbf{l}}(h)$. ✠

Corollary A.5.1. *A right adjoint functor preserves monomorphism.*

Proof. Let $\mathrm{G}{:}\mathbf{C}{\to}\mathbf{D}$ be a right adjoint and let $f \in \overline{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ be a monomorphism. Then

$$\begin{array}{ccc} \mathbf{c}_1 & \xrightarrow{\mathrm{id}_{\mathbf{c}_1}} & \mathbf{c}_1 \\ {\scriptstyle \mathrm{id}_{\mathbf{c}_1}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{c}_1 & \xrightarrow{f} & \mathbf{c}_2 \end{array} \tag{A.137}$$

is a pull-back, and by Proposition A.5.8 so is

$$\begin{array}{ccc} \mathrm{F}(\mathbf{c}_1) & \xrightarrow{\mathrm{F}(\mathrm{id}_{\mathbf{c}_1})} & \mathrm{F}(\mathbf{c}_1) \\ {\scriptstyle \mathrm{F}(\mathrm{id}_{\mathbf{c}_1})}\downarrow & & \downarrow{\scriptstyle \mathrm{F}(f)} \\ \mathrm{F}(\mathbf{c}_1) & \xrightarrow{\mathrm{F}(f)} & \mathbf{c}_2 \end{array} \tag{A.138}$$

so $\mathrm{F}(f)$ is a monomorphism. ✠

Proposition A.5.9. *A left adjoint functor preserves direct limits.*

Proof. Analogous to proof of Proposition A.5.8 ✠

Corollary A.5.2. *A left adjoint functor preserves epimorphism.*

Proof. Analogous to proof of Corollary A.5.1 ✠

Definition A.5.4. Let $\mathbf{C}$ be a category, $\mathrm{S} \subseteq \mathcal{O}(\mathbf{C})$. The ***immersion functor of*** S **in** $\mathbf{C}$ is the functor $\mathrm{I}^{\mathrm{S}}{:}S^{*}{\to}\mathbf{C}$ defined on objects by

$$\mathrm{I}^{\mathrm{S}}(\mathbf{c}) = \mathbf{c} \tag{A.139}$$

and on morphisms by

$$\mathrm{I}^{\mathrm{S}}(\mathrm{id}_{\mathbf{c}}) = \mathrm{id}_{\mathbf{c}}\,. \tag{A.140}$$

Definition A.5.5. We say that a category $\mathbf{C}$ ***satisfies the Solution Set Condition*** if there is a small subset S of $\mathcal{O}(\mathbf{C})$ such that for every object $\mathbf{c}$ of $\mathbf{C}$ there is an object $\mathbf{c}' \in \mathrm{S}$ and a morphism $f : \mathbf{c}' \to \mathbf{c}$. The set S is called the ***solution set of*** $\mathbf{C}$.

Theorem A.5.1. *Let* $\mathbf{C}$ *be a small-inverse-complete category with small hom-sets. Then* $\mathbf{C}$ *has an initial object if and only if it satisfies the Solution Set Condition.*

Proof. Suppose $\mathbf{C}$ has an initial object $\mathbf{i}$. Then $\{\mathbf{i}\}$ is a solution set of $\mathbf{C}$. Suppose S is an initial set of $\mathbf{C}$. Let $\mathrm{I^S}$ be the immersion functor of S in $\mathbf{C}$. Since $\mathbf{C}$ is small complete the inverse product $(\mathbf{p}, \pi)$ of $\mathrm{I^S}$ exists. Since the set $\overline{\mathbf{C}}(\mathbf{p}, \mathbf{p})$ is small and $\mathbf{C}$ is small-complete an equaliser $e : \mathbf{i} \to \mathbf{p}$ of it exists. For $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ there is an object $\mathbf{c}^* \in \mathrm{S}$ and a morphism $f : \mathbf{c}^* \to \mathbf{c}$, and thus also a morphism $f \circ \pi_{\mathbf{c}^*} \circ e : \mathbf{i} \to \mathbf{c}$. Suppose there are two morphisms $g_1, g_2 : \mathbf{i} \to \mathbf{c}$, then there exists an equaliser of them $h : \mathbf{u} \to \mathbf{i}$. By construction of $\mathbf{p}$ there is a morphism $s : \mathbf{p} \to \mathbf{u}$, thus $e \circ h \circ s \in \overline{\mathbf{C}}(\mathbf{p}, \mathbf{p})$, and since e is an equaliser of $\overline{\mathbf{C}}(\mathbf{p}, \mathbf{p})$ we have $e \circ h \circ s \circ e = \mathrm{id}_{\mathbf{p}} \circ e = e \circ \mathrm{id}_{\mathbf{i}}$ whence, since e is a monomorphism, $h \circ s \circ e = \mathrm{id}_{\mathbf{i}}$. Thus h has a right inverse whence, since it is a monomorphism, it is an isomorphism, which yields $g_1 = g_2$. ✠

Definition A.5.6. Let $\mathrm{F} : \mathbf{C} \to \mathbf{D}$ be a functor, $\mathbf{d} \in \mathcal{O}(\mathbf{D})$. We call ***projection functor*** F***-under*** $\mathbf{d}$ the functor $\mathrm{Q^{\mathbf{d},F}}$ defined on object by $\mathrm{Q^{\mathbf{d},F}}((\mathbf{c}, u)) = \mathbf{c}$ and on morphisms by $\mathrm{Q^{\mathbf{d},F}}(f) = f$. We call ***projection functor*** F***-over*** $\mathbf{d}$ the functor $\mathrm{Q^{F,\mathbf{d}}}$ defined on object by $\mathrm{Q^{F,\mathbf{d}}}((u, \mathbf{c})) = \mathbf{c}$ and on morphisms by $\mathrm{Q^{\mathbf{d},F}}(f) = f$

Theorem A.5.2. *If the functor* $\mathrm{F} : \mathbf{C} \to \mathbf{D}$ *preserves small inverse limits then for every* $\mathbf{d} \in \mathcal{O}(\mathbf{D})$ *the projection functor* F*-under* $\mathbf{d}$ *creates all small inverse limits.*

Proof. Let $\mathbf{J}$ be a small category, $\mathrm{H} : \mathbf{J} \to \mathbf{C}$ and $(\mathbf{l}, \lambda)$ an inverse limit of $\mathrm{Q^{\mathbf{d},F}H}$. Then $(\mathrm{F}(\mathbf{l}), \mathrm{F}\lambda)$ is an inverse limit of $\mathrm{FQ^{\mathbf{d},F}H}$. For $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ set $\mathrm{H}(\mathbf{j}) = (\mathbf{c}_{\mathbf{j}}, \mu_{\mathbf{j}})$, and the $\mu_{\mathbf{j}}$ are the components of a natural transformation $\mu : \mathrm{C}_{\mathbf{d}}^{\mathbf{C},\mathbf{D}} \to \mathrm{F}$. Thus $(\mathbf{d}, \mu)$ is an inverse target for F, and there is a unique morphism $f : \mathbf{d} \to \mathrm{F}(\mathbf{l})$ such that

$$\mu = \mathrm{F}\lambda \circ \gamma_f^{\mathbf{C},\mathbf{D}}. \tag{A.141}$$

Since $\mathrm{F}\lambda_{\mathbf{j}} : \mathrm{F}(\mathbf{l}) \to \mathrm{FQ^{\mathbf{d},F}H}(\mathbf{j})$ and $\mathrm{FQ^{\mathbf{d},F}H}(\mathbf{j}) = \mathrm{F}(\mathbf{c}_{\mathbf{j}})$, then from Eq. A.141 follows that $\mathrm{F}\lambda_{\mathbf{j}} \in \overline{(d \downarrow F)}((\mathbf{l}, f), \mathrm{H}(\mathbf{j}))$, and $((\mathbf{l}, f), \mathrm{F}\lambda)$ is an inverse target of H.
If $((\mathbf{m}, g), \tau)$ is another inverse target of H, then $(\mathbf{m}, \tau)$ is an inverse target of $\mathrm{Q^{\mathbf{d},F}H}$, thus there is a unique morphism $h : \mathbf{m} \to \mathbf{l}$ such that

$$\tau = \lambda \circ \gamma_h^{\mathbf{J},\mathbf{C}}. \tag{A.142}$$

Let's show that $h \in \overline{(d \downarrow F)}((\mathbf{m}, g), (\mathbf{l}, f))$. From Eq. A.142 follows $\mathrm{F}\tau = \mathrm{F}\lambda \circ \gamma_{\mathrm{F}(h)}^{\mathbf{J},\mathbf{C}}$ and

$$\mathrm{F}\lambda \circ \gamma_{\mathrm{F}(h) \circ g}^{\mathbf{J},\mathbf{C}} = \mathrm{F}\lambda \circ \gamma_{\mathrm{F}(h)}^{\mathbf{J},\mathbf{C}} \circ \gamma_g^{\mathbf{J},\mathbf{C}} = \mathrm{F}\tau \circ \gamma_g^{\mathbf{J},\mathbf{C}} = \mu \tag{A.143}$$

and the uniqueness of f in Eq. A.141 yields $\mathrm{F}(h)\circ g = f$.
Suppose there is also $k:(\mathbf{m},g)\to(\mathbf{l},f)$ such that $\tau=\lambda\circ\gamma_k^{\mathbf{J},\mathbf{C}}$, then $k\in\overline{\mathbf{C}}(\mathbf{m},\mathbf{l})$ and the uniqueness of h in Eq. A.142 yields $k=h$.
Thus $((\mathbf{l},f),\mathrm{F}\lambda)$ is a limit of H. ✠

Theorem A.5.3 (Freyd Adjoint Functor Theorem). *Let* $\mathbf{D}$ *be a small-inverse-complete category with small hom-sets. A functor* $\mathrm{F}:\mathbf{D}\to\mathbf{C}$ *has a left adjoint if and only if it preserve all small inverse limits and for every object* $\mathbf{c}\in\mathcal{O}\,\mathbf{c}$ *the category* $(c\downarrow G)$ *satisfies a Solution Set Condition.*

Proof. If G has a left adjoint then it preserves all limits, in particular all small ones. If F is the left adjoint of G and η is the unit of the adjunction then $\{(\mathrm{F}(\mathbf{c}),\eta_{\mathbf{c}})\}$ is a solution set for $(c\downarrow G)$. Indeed, if ε is the counit of the adjunction then for $(\mathbf{d},u)\in\mathcal{O}((c\downarrow G))$ $\varepsilon_{\mathbf{d}}\circ\mathrm{F}(u)\in\overline{\mathbf{D}}(\mathrm{F}(\mathbf{c}),\mathbf{d})$ and $\mathrm{G}(\varepsilon_{\mathbf{d}}\circ\mathrm{F}(u))\circ\eta_{\mathbf{c}}=u$, thus $\varepsilon_{\mathbf{d}}\circ\mathrm{F}(u)\in\overline{(c\downarrow G)}((\mathrm{F}(\mathbf{c}),\eta_{\mathbf{c}}),(\mathbf{d},u))$.
If G preserves all small inverse limits then by Theorem A.5.2 for every $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ the category $(c\downarrow G)$ is small-inverse-complete.
Since $\mathbf{D}$ has small hom-sets for every $\mathbf{c}\in\mathcal{O}(\mathbf{C})$ the category $(c\downarrow G)$ also has small hom-sets.
If for every object $\mathbf{c}\in\mathcal{O}(\mathbf{c})$ the category $(c\downarrow G)$ satisfies a Solution Set Condition, then, since it is small-inverse-complete and has small hom-sets, by Theorem A.5.1 it has an initial object, thus by Proposition A.2.5 there is a universal arrow from $\mathbf{c}$ to G. By Proposition A.5.4 G has a left adjoint. ✠

Proposition A.5.10. *Every functor from* $\mathbf{J}$ *to* $\mathbf{C}$ *has a direct limit if and only if* $\Delta_{\mathbf{C}}^{\mathbf{J}}$ *has a left adjoint.*

Proof. Suppose $\Delta_{\mathbf{C}}^{\mathbf{J}}$ has a left adjoint $\mathcal{L}$ and let ϕ be the adjunction's natural isomorphism. Then by Proposition A.2.16 for $\mathrm{F}\in\mathcal{O}(\mathbf{C}^{\mathbf{J}})$ $(\mathcal{L}(\mathrm{F}),\phi_{\mathcal{L}(\mathrm{F}),\mathcal{L}(\mathrm{F})}(\mathrm{id}_{\mathcal{L}(\mathrm{F})}))$ is a direct limit for F.
If every $\mathrm{F}\in\mathcal{O}(\mathbf{C}^{\mathbf{J}})$ has a limit $(\mathbf{l}_{\mathrm{F}},\lambda_{\mathrm{F}})$, then this is a universal arrow from F to $\Delta_{\mathbf{C}}^{\mathbf{J}}$, thus by Proposition A.5.4 $\Delta_{\mathbf{C}}^{\mathbf{J}}$ has a left adjoint. ✠

A.6 Biproducts

Proposition A.6.1. *Let* $\mathbf{A}$ *be an ab-category. If* $(\mathbf{c},p_i)$ *is an inverse product of the objects* $\{\mathbf{a}_1\ldots\mathbf{a}_i\}$ *then* $(\mathbf{c},i_i)$ *is a direct product of* $\{\mathbf{a}_1\ldots\mathbf{a}_i\}$*, where the morphisms* i_i *are defined by*

$$p_i i_j=\begin{cases}\mathrm{id}_{\mathbf{a}_i},\ \textit{if } i=j\\ \mathbf{0}_{\mathbf{a}_i}^{\mathbf{a}_j},\ \textit{if } i\neq j.\end{cases} \tag{A.144}$$

If $(\mathbf{d}, i_i)$ *is a direct product of the objects* $\{\mathbf{a}_1 \ldots \mathbf{a}_i\}$ *then* $(\mathbf{d}, p_i)$ *is an inverse product of* $\{\mathbf{a}_1 \ldots \mathbf{a}_i\}$*, where the morphisms* p_i *are defined by*

$$p_i i_j = \begin{cases} \mathrm{id}_{\mathbf{a}_i}, \textit{if } i = j \\ \mathbf{0}^{\mathbf{a}_j}_{\mathbf{a}_i}, \textit{ if } i \neq j. \end{cases} \tag{A.145}$$

Proof. Suppose $(\mathbf{c}, p_i)$ is an inverse product of the objects $\{\mathbf{a}_1 \ldots \mathbf{a}_i\}$ and let i_i for $i = 1 \cdots n$ be defined by Equation (A.144). We have $i_1 p_1 + \cdots + i_n p_n = \mathrm{id}_{\mathbf{c}}$, indeed

$$p_j(i_1 p_1 + \cdots + i_n p_n) = p_j. \tag{A.146}$$

Let $\mathbf{e} \in \mathcal{O}(\mathbf{A})$ and $f_i : \mathbf{a}_i \to \mathbf{e}$ for $i = 1 \cdots n$. Let

$$h = f_i p_i + \ldots + f_n p_n. \tag{A.147}$$

Then $h i_i = f_i$ for $i = 1 \cdots n$. If $k : \mathbf{c} \to \mathbf{e}$ such that $k i_i = f_i$ for $i = 1 \cdots n$, then

$$k i_1 p_1 + \cdots + k i_n p_n = f p_1 + \cdots + f p_n \tag{A.148}$$

but

$$k i_1 p_1 + \cdots + k i_n p_n = k(i_1 p_1 + \cdots + i_n p_n) = k\, \mathrm{id}_{\mathbf{c}} = k \tag{A.149}$$

thus $k = h$.
The proof of the second part is analogous. ✠

Definition A.6.1. Let $\mathbf{A}$ be a category with zero morphisms, $\mathbf{a}_1, \ldots, \mathbf{a}_n$ objects of $\mathbf{A}$. An object $\mathbf{c}$ is a ***biproduct of*** $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if there are morphisms $p_i : \mathbf{c} \to \mathbf{a}_i$ and $i_i : \mathbf{a}_i \to \mathbf{c}$ for $i = 1, \ldots, n$ such that

$$p_i i_j = \begin{cases} \mathrm{id}_{\mathbf{a}_i}, \text{if } i = j \\ \mathbf{0}^{\mathbf{a}_j}_{\mathbf{a}_i}, \text{ if } i \neq j \end{cases} \tag{A.150}$$

and

- $\mathbf{c}$ is an inverse product of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ with projecions p_i
- $\mathbf{c}$ is a direct product of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ with injecions i_i.

The morphisms p_i are called the ***projections*** of $\mathbf{c}$, the morphisms i_i are called the ***injections*** of $\mathbf{c}$.

Proposition A.6.2. *Let* $\mathbf{A}$ *be an ab-category,* $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathcal{O}(\mathbf{A})$*,* $\mathbf{c} \in \mathcal{O}(\mathbf{A})$ *for which there are morphisms* $p_i : \mathbf{c} \to \mathbf{a}_i$ *and* $i_i : \mathbf{a}_i \to \mathbf{c}$ *for* $i = 1, \ldots, n$ *such that*

$$p_i i_j = \begin{cases} \mathrm{id}_{\mathbf{a}_i}, \textit{if } i = j \\ \mathbf{0}^{\mathbf{a}_j}_{\mathbf{a}_i}, \textit{ if } i \neq j \end{cases} \tag{A.151}$$

and

$$i_1 p_1 + \cdots + i_n p_n = \mathrm{id}_{\mathbf{c}}\,. \tag{A.152}$$

Then $\mathbf{c}$ *is a biproduct of* $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathcal{O}(\mathbf{A})$ *with projections* p_i *and injections* i_i*.*

Proof. By Proposition A.6.1 it is sufficient to prove that $\mathbf{c}$ is an inverse product of $\mathbf{a}_1, \dots, \mathbf{a}_n$ with projecions p_i.
Let $\mathbf{d} \in \mathcal{O}(\mathbf{A})$ and $q_1 : \mathbf{d} \to \mathbf{a}_1$ for $i = 1, \dots, n$. Define $h : \mathbf{d} \to \mathbf{c}$ by

$$h = \sum_{i=1}^{n} i_i q_i. \tag{A.153}$$

Then

$$p_j h = p_j \sum_{i=1}^{n} i_i q_i = \sum_{i=1}^{n} p_j i_i q_i = p_j i_j q_j = \mathrm{id}_{\mathbf{a}_j} q_j = q_j. \tag{A.154}$$

If also $k : \mathbf{d} \to \mathbf{c}$ is such that $p_i k = q_i$ for $i = 1, \dots, n$. Then

$$k = \mathrm{id}_{\mathbf{c}} k = \sum_{i=1}^{n} i_i p_i k = \sum_{i=1}^{n} i_i p_i h = \mathrm{id}_{\mathbf{c}} h = h. \tag{A.155}$$

✠

Proposition A.6.3. *Let* $\mathbf{A}$ *be an ab-category,* $\mathbf{a}_1, \dots, \mathbf{a}_n$ *objects of* $\mathbf{A}$*,* $\mathbf{c}$ *a biproduct of* $\mathbf{a}_1, \dots, \mathbf{a}_n$ *with projections* p_i *and injections* i_i*. Then* $\mathbf{c}$ *is an inverse product of* $\mathbf{a}_1, \dots, \mathbf{a}_n$ *with projections* p_i *and a direct product of* $\mathbf{a}_1, \dots, \mathbf{a}_n$ *with injections* i_i*.*

Proof. Routine check. ✠

Notation A.6.1. We will write

$$\bigodot_{i}^{n} \mathbf{a}_i \tag{A.156}$$

or

$$\mathbf{a}_1 \odot \cdots \odot \mathbf{a}_n \tag{A.157}$$

or, when there is no risk of ambiguity

$$\bigodot a_i, \odot a_i, \tag{A.158}$$

for the isomorphism class of the biproducts of $\mathbf{a}_1, \dots, \mathbf{a}_n$, and also for a specific member of it when this will not generate any ambiguity. We will write

$$p^{\bullet}, i^{\bullet} \tag{A.159}$$

respectively for the projections and injections of the biproduct indicated by $\bullet$.

Proposition A.6.4. *Let* $\mathbf{A}$ *be a category with zero morphisms,* $\mathbf{a}_1, \dots, \mathbf{a}_n$*,* $\mathbf{b}_1, \dots, \mathbf{b}_n$ *objects of* $\mathbf{A}$*,* $f_i : \mathbf{a}_i \to \mathbf{b}_i$ *for* $i = 1, \dots, n$*, and suppose that* $\odot \mathbf{a}_i$ *and* $\odot \mathbf{b}_i$ *exist. Then there is a unique map* $\odot f_i : \odot a_i \to \odot b_i$ *satisfying for* $j = 1, \dots, n$

$$\begin{aligned} p_j^{\odot \mathbf{b}_i} \odot f_i &= f_j p_j^{\odot \mathbf{a}_i} \\ \odot f_i i_j^{\odot \mathbf{a}_i} &= i_j^{\odot \mathbf{b}_i} f_j. \end{aligned} \tag{A.160}$$

If $\mathbf{A}$ *is an ab-category then*

$$\odot f_i = \sum_{j=1}^{n} i_j^{\odot \mathbf{b}_i} f_j p_j^{\odot \mathbf{a}_i}. \tag{A.161}$$

Proof. That the (A.160) define a unique map $\odot f_i : \odot a_i \to \odot b_i$ is a consequence of being $\odot a_i$ an inverse product with projections $p^{\odot a_i}$, and $\odot b_i$ a direct product with injections $i^{\odot b_i}$. From the first of (A.160) we get

$$\sum_{j=1}^{n} i_j^{\odot \mathbf{b}_i} p_j^{\odot \mathbf{b}_i} \odot f_i = \sum_{j=1}^{n} i_j^{\odot \mathbf{b}_i} f_j p_j^{\odot \mathbf{a}_i} \tag{A.162}$$

but

$$\sum_{j=1}^{n} i_j^{\odot \mathbf{b}_i} p_j^{\odot \mathbf{b}_i} \odot f_i = \mathrm{id}_{\odot \mathbf{b}_i} \odot f_i = \odot f_i. \tag{A.163}$$

✠

Definition A.6.2. For a category with zero morphisms $\mathbf{C}$ and $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ the maps

$$\delta_{\mathbf{c}} : \mathbf{c} \to \mathbf{c} \prod \mathbf{c} \tag{A.164}$$

$$\delta^{\mathbf{c}} : \mathbf{c} \coprod \mathbf{c} \to \mathbf{c} \tag{A.165}$$

defined respectively by

$$p_1^{\mathbf{c} \prod \mathbf{c}} \delta_{\mathbf{c}} = p_2^{\mathbf{c} \prod \mathbf{c}} \delta_{\mathbf{c}} = \mathrm{id}_{\mathbf{c}} \tag{A.166}$$

$$\delta^{\mathbf{c}} i_1^{\mathbf{c} \coprod \mathbf{c}} = \delta^{\mathbf{c}} i_2^{\mathbf{c} \coprod \mathbf{c}} = \mathrm{id}_{\mathbf{c}} \tag{A.167}$$

are called respectively the ***diagonal map to*** $\mathbf{c} \prod \mathbf{c}$ and the ***diagonal map from*** $\mathbf{c} \coprod \mathbf{c}$.

Lemma A.6.1. *In an ab-category the biproduct is bilinear with respect to composition. That is, if* $f_1 : \mathbf{a}_1 \to \mathbf{b}_1$, $g_1 : \mathbf{b}_1 \to \mathbf{c}_1$, $f_2 : \mathbf{a}_2 \to \mathbf{b}_2$, $g_2 : \mathbf{b}_2 \to \mathbf{c}_2$, *and* $\mathbf{a}_1 \odot \mathbf{a}_2$, $\mathbf{b}_1 \odot \mathbf{b}_2$, $\mathbf{c}_1 \odot \mathbf{c}_2$ *exist, then* $(g_1 \odot g_2) \circ (f_1 \odot f_2) = (g_1 \circ f_1) \odot (g_2 \circ f_2)$.

Proof. We have

$$\begin{aligned}
(g_1 \odot g_2) \circ (f_1 \odot f_2) = {} & (i_1^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_1 \circ p_1^{\mathbf{b}_1 \odot \mathbf{b}_2} + i_2^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_2 \circ p_2^{\mathbf{b}_1 \odot \mathbf{b}_2}) \circ \\
& \circ (i_1^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ f_1 \circ p_1^{\mathbf{a}_1 \odot \mathbf{a}_2} + i_2^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ f_2 \circ p_2^{\mathbf{a}_1 \odot \mathbf{a}_2}) = \\
= {} & (i_1^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_1 \circ p_1^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ i_1^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ f_1 \circ p_1^{\mathbf{a}_1 \odot \mathbf{a}_2}) + \\
& + (i_2^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_2 \circ p_2^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ i_1^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ f_1 \circ p_1^{\mathbf{a}_1 \odot \mathbf{a}_2}) + \\
& + (i_1^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_1 \circ p_1^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ i_2^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ f_2 \circ p_2^{\mathbf{a}_1 \odot \mathbf{a}_2}) + \\
& + (i_2^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_2 \circ p_2^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ i_2^{\mathbf{b}_1 \odot \mathbf{b}_2} \circ f_2 \circ p_2^{\mathbf{a}_1 \odot \mathbf{a}_2}) = \\
= {} & (i_1^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_1 \circ f_1 \circ p_1^{\mathbf{a}_1 \odot \mathbf{a}_2}) + (i_2^{\mathbf{c}_1 \odot \mathbf{c}_2} \circ g_2 \circ f_2 \circ p_2^{\mathbf{a}_1 \odot \mathbf{a}_2}) = \\
= {} & (g_1 \circ f_1) \odot (g_2 \circ f_2).
\end{aligned}$$

✠

Remark A.6.1. The result of Lemma A.6.1 can of course be extended to any number of operands, that is

$$\bigodot_{i=1}^{n}(f_i^1 \circ \cdots \circ f_i^m) = \bigodot_{i=1}^{n} f_i^1 \circ \cdots \circ \bigodot_{i=1}^{n} f_i^m \tag{A.168}$$

Lemma A.6.2. *For $f_1 : \mathbf{a} \to \mathbf{b}$, $f_1 : \mathbf{a} \to \mathbf{b}$ in an ab-category*

$$f_1 + f_2 = \delta^{\mathbf{b}}(f_1 \odot f_2)\delta_{\mathbf{a}}. \tag{A.169}$$

Proof.

$$\begin{aligned}\delta^{\mathbf{b}}(f_1 \odot f_2)\delta_{\mathbf{a}} &= \delta^{\mathbf{b}}(i_1^{\mathbf{b}\odot\mathbf{b}} f_1 p_1^{\mathbf{a}\odot\mathbf{a}} + i_2^{\mathbf{b}\odot\mathbf{b}} f_2 p_2^{\mathbf{a}\odot\mathbf{a}})\delta_{\mathbf{a}} = \\ &= \delta^{\mathbf{b}}(i_1^{\mathbf{b}\odot\mathbf{b}} f_1 \operatorname{id}_{\mathbf{a}} + i_2^{\mathbf{b}\odot\mathbf{b}} f_2 \operatorname{id}_{\mathbf{a}}) = \\ &= \operatorname{id}_{\mathbf{b}} f_1 + \operatorname{id}_{\mathbf{b}} f_2 = \\ &= f_1 + f_2.\end{aligned}$$

✠

Definition A.6.3. A functor $\mathrm{F}:\mathbf{A}\to\mathbf{B}$ betewen ab-categories is ***additive*** if for any two morphisms $f_1 : \mathbf{a} \to \mathbf{b}$, $f_2 : \mathbf{a} \to \mathbf{b}$

$$\mathrm{F}(f_1 + f_2) = \mathrm{F}(f_1) + \mathrm{F}(f_2). \tag{A.170}$$

!!! Maybe add here def of ”preserve biproducts”? !!!

Proposition A.6.5. *If $\mathbf{A}$ and $\mathbf{B}$ are ab-categories and $\mathbf{A}$ has binary biproducts then a functor $\mathrm{F}:\mathbf{A}\to\mathbf{B}$ is additive f and only if it preserves biproducts.*

Proof. If F is additive then it is easy to check that for $\mathbf{a}_1 \in \mathcal{O}(\mathbf{A})$, $\mathbf{a}_2 \in \mathcal{O}(\mathbf{A})$, $\mathrm{F}(\mathbf{a}_1 \odot \mathbf{a}_2)$ is a biproduct of $\mathrm{F}(\mathbf{a}_1)$ and $\mathrm{F}(\mathbf{a}_2)$ with projections $\mathrm{F}(p^{\mathbf{a}_1\odot\mathbf{a}_2})$ and injections $\mathrm{F}(i^{\mathbf{a}_1\odot\mathbf{a}_2})$.
Suppose F preserves biproducts and let $f_1 : \mathbf{a} \to \mathbf{b}$, $f_1 : \mathbf{a} \to \mathbf{b}$. Since F preserves biproducts $\mathrm{F}(\delta_{\mathbf{a}}) = \delta_{\mathrm{F}(\mathbf{a})}$ and $\mathrm{F}(\delta^{\mathbf{b}}) = \delta^{\mathrm{F}(\mathbf{b})}$, thus

$$\mathrm{F}(f_1 + f_2) = \mathrm{F}(\delta^{\mathbf{b}}(f_1 \odot f_2)\delta_{\mathbf{a}}) = \delta^{\mathrm{F}(\mathbf{b})}\left(\mathrm{F}(f_1) \odot \mathrm{F}(f_2)\right)\delta_{\mathrm{F}(\mathbf{a})} = \mathrm{F}(f_1) + \mathrm{F}(f_2). \tag{A.171}$$

✠

Proposition A.6.6. *Let $\mathbf{C}$ be a category, $\mathbf{A}$ an ab-category with all biproducts. Then for $n \in \mathbb{N}$ there is a functor $\bigodot^n:\mathbf{C}^n\to\mathbf{A}$ such that for $\mathbf{a}_i \in \mathcal{O}(\mathbf{A})$, $f_i \in \mathcal{M}(\mathbf{A})$, $i = 1, \dots, n$*

- $\bigodot^n(\mathbf{a}_1, \dots, \mathbf{a}_n) = \bigodot_{i=1}^n \mathbf{a}_i$
- $\bigodot^n(f_1, \dots, f_n) = \bigodot_{i=1}^n f_i$.

Proof. Let $\mathbf{a}_i \in \mathcal{O}(\mathbf{A})$, $i = 1, \ldots, n$. Then

$$\bigodot^{n}(\mathrm{id}_{\mathbf{a}_1}, \ldots, \mathrm{id}_{\mathbf{a}_n}) = \sum_{j=1}^{n} i_j^{\odot a_i} \,\mathrm{id}_{\mathbf{a}_j}\, p_j^{\odot \mathbf{a}_i} =$$

$$= \sum_{j=1}^{n} i_j^{\odot a_i} p_j^{\odot \mathbf{a}_i} =$$

$$= \mathrm{id}_{\odot \mathbf{a}_i}\,.$$

Let $(f_1, f_2) : (\mathbf{a}_1, \mathbf{a}_2) \to (\mathbf{b}_1, \mathbf{b}_2)$, $(g_1, g_2) : (\mathbf{b}_1, \mathbf{b}_2) \to (\mathbf{c}_1, \mathbf{c}_2)$. Then by Lemma A.6.1 and Remark A.6.1

$$\bigodot^{n}(g_1 \circ f_1, \ldots, g_n \circ f_n) = \bigodot_{i=1}^{n}(g_i \circ f_i) = \bigodot_{i=1}^{n}(g_i) \circ \bigodot_{i=1}^{n}(f_i) = \tag{A.172}$$

$$= \bigodot^{n}(g_1, \ldots, g_n) \circ \bigodot^{n}(f_1, \ldots, f_n). \tag{A.173}$$

✠

Proposition A.6.7. *Let* $\mathbf{D}$ *be a category with n-order biproducts. Then for every category* $\mathbf{C}$ *the category* $\mathbf{D}^{\mathbf{C}}$ *also has n-order biproducts.*

Proof. Let $\mathrm{F}_1, \ldots, \mathrm{F}_n \in \mathcal{O}(\mathbf{D}^{\mathbf{C}})$. Let's show that there is a functor $\odot \mathrm{F}_1 \in \mathcal{O}(\mathbf{D}^{\mathbf{C}})$ defined by:

- $(\odot \mathrm{F}_i)(\mathbf{c}) = \odot \mathrm{F}_i(\mathbf{c})$ for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$
- $(\odot \mathrm{F}_i)(f) = \odot \mathrm{F}_i(f)$ for $f \in \mathcal{M}(\mathbf{C})$.

For $\mathbf{c} \in \mathcal{O}(\mathbf{C})$

$$(\odot \mathrm{F}_i)(\mathrm{id}_{\mathbf{C}}) = \odot \mathrm{F}_i(\mathrm{id}_{\mathbf{c}}) = \odot\, \mathrm{id}_{\mathrm{F}(\mathbf{c})} = \mathrm{id}_{\odot \mathrm{F}(\mathbf{c})}\,. \tag{A.174}$$

For composable morphisms $f, g \in \mathcal{M}(\mathbf{C})$ by Lemma A.6.1

$$(\odot \mathrm{F}_i)(g \circ f) = \odot \mathrm{F}_i(g \circ f) = \odot(\mathrm{F}_i(g) \circ \mathrm{F}_i(f)) = \odot \mathrm{F}_i(g) \circ \odot \mathrm{F}_i(f). \tag{A.175}$$

Let's show that $\odot \mathrm{F}_i$ is a biproduct of $\mathrm{F}_1, \ldots, \mathrm{F}_n$. It will suffice to show that it is an inverse product of $\mathrm{F}_1, \ldots, \mathrm{F}_n$. Let's show that there are natural transformation $p_j^{\odot \mathrm{F}_i} : \odot\, \mathrm{F}_i \to \mathrm{F}_j$ for $j = 1, \ldots, n$ defined for $\mathbf{c} \in \mathcal{O}(\mathbf{C})$ by

$$(p_j^{\odot \mathrm{F}_i})_{\mathbf{c}} = p_j^{\odot \mathrm{F}_i(\mathbf{c})}. \tag{A.176}$$

Let $\mathbf{c}, \mathbf{d} \in \mathcal{O}(\mathbf{C})$, $f \in \overline{\mathbf{C}}(\mathbf{c}, \mathbf{d})$. The diagram

$$\begin{array}{ccc} (\odot \mathrm{F}_i)(\mathbf{c}) & \xrightarrow{(p_j^{\odot \mathrm{F}_i})_{\mathbf{c}}} & \mathrm{F}_j(\mathbf{c}) \\ \big\downarrow{\scriptstyle (\odot \mathrm{F}_i)(f)} & & \big\downarrow{\scriptstyle \mathrm{F}_j(f)} \\ (\odot \mathrm{F}_i)(\mathbf{d}) & \xrightarrow{(p_j^{\odot \mathrm{F}_i})_{\mathbf{d}}} & \mathrm{F}_j(\mathbf{d}) \end{array} \tag{A.177}$$

commutes because

$$\mathrm{F}_j(f) \circ (p_j^{\odot \mathrm{F}_i})_{\mathbf{c}} = \mathrm{F}_j(f) \circ p_j^{\odot \mathrm{F}_i(\mathbf{c})}$$
$$(p_j^{\odot \mathrm{F}_i})_{\mathbf{d}} \circ (\odot \mathrm{F}_i)(f) = p_j^{\odot \mathrm{F}_i(\mathbf{d})} \circ \odot \mathrm{F}_i(f)$$

and

$$\mathrm{F}_j(f) \circ p_j^{\odot \mathrm{F}_i(\mathbf{c})} = p_j^{\odot \mathrm{F}_i(\mathbf{d})} \circ \odot \mathrm{F}_i(f) \tag{A.178}$$

by the definition of $\odot \mathrm{F}_i(f)$. ✠

Lemma A.6.3. *Let* $\mathbf{A}$ *be an ab-category with finite biproducts,* $\mathrm{F}_i \in \mathcal{O}(\mathbf{A}^{\mathbf{C}})$ *for* $i = 1, \ldots, n$ *and suppose that* $(\mathbf{c}_i, \pi_i)$ *is an inverse limit for* F_i *for* $i = 1, \ldots, n$*. Then* $(\odot \mathbf{c}_i, \odot \pi_i)$ *is an inverse limit for* $\odot \mathrm{F}_i$*.*

Proof. Note that $\odot \mathrm{C}_{\mathbf{c}_i}^{\mathbf{C},\mathbf{A}} = \mathrm{C}_{\odot \mathbf{c}_i}^{\mathbf{C},\mathbf{A}}$, therefore the biproduct $\odot \pi_i$ is a natural transformation

$$\odot \pi_i : \mathrm{C}_{\odot \mathbf{c}_i}^{\mathbf{C},\mathbf{A}} \to \odot \mathrm{F}_i \tag{A.179}$$

so $(\odot \mathbf{c}_i, \odot \pi_i)$ is an inverse target for $\odot \mathrm{F}_i$. If $(\mathbf{d}, \mathrm{h})$ is an inverse target for $\odot \mathrm{F}_i$, for $i = 1, \ldots, n$ set

$$\mathrm{h}_i = p_i^{\odot \mathrm{F}i} \circ \mathrm{h}; \tag{A.180}$$

these are natural transformations $\mathrm{h}_i : \mathrm{C}_{\mathbf{d}}^{\mathbf{C},\mathbf{A}} \to \mathrm{F}_i$ that factor uniquely as:

$$\mathrm{h}_i = \pi_i \circ k_i \tag{A.181}$$

where k_i for $i = 1, \ldots, n$ are natural transformations

$$k_i : \mathrm{C}_{\mathbf{d}}^{\mathbf{C},\mathbf{A}} \to \mathrm{C}_{\mathbf{c}_i}^{\mathbf{C},\mathbf{A}}. \tag{A.182}$$

Set

$$\mathrm{h}^* = \sum i_i^{\mathrm{C}_{\odot \mathbf{c}_i}^{\mathbf{C},\mathbf{D}}} \circ k_i.$$

Then $\odot \pi_i \circ \mathrm{h}^* = \mathrm{h}$, indeed

$$\begin{aligned}\odot \pi_i \circ \mathrm{h}^* &= \odot \pi_i \circ \sum i_j^{\mathrm{C}_{\odot \mathbf{c}_i}^{\mathbf{C},\mathbf{D}}} \circ \mathrm{k}_j = \sum \odot \pi_i \circ i_j^{\mathrm{C}_{\odot \mathbf{c}_i}^{\mathbf{C},\mathbf{D}}} \circ \mathrm{k}_j = \sum i_j^{\odot \mathrm{F}_i} \circ \pi_j \circ \mathrm{k}_j = \\ &= \sum i_j^{\odot \mathrm{F}_i} \circ \mathrm{h}_j = \sum i_j^{\odot \mathrm{F}_i} \circ p_j^{\odot \mathrm{F}_i} \mathrm{h} = \mathrm{id}_{\odot \mathrm{F}_i} \mathrm{h} = \mathrm{h}.\end{aligned}$$

If $\mathrm{l} : \mathrm{C}_{\mathbf{d}}^{\mathbf{C},\mathbf{A}} \to \mathrm{C}_{\odot \mathrm{F}_i}^{\mathbf{C},\mathbf{A}}$ is such that $\odot \pi_i \circ \mathrm{l} = \mathrm{h}$, then

$$\mathrm{h} = \sum_i (i_i^{\odot \mathrm{F}_j} \circ \pi_i \circ p_i^{\mathrm{C}_{\odot \mathbf{c}_j}^{\mathbf{C},\mathbf{A}}}) \circ \mathrm{l}$$

thus

$$h_i = p_i^{\odot F_j} h = \pi_i \circ p_i^{C^{\mathbf{C},\mathbf{A}}_{\odot \mathbf{c}_j}} \circ l \tag{A.183}$$

which yields

$$p_i^{C^{\mathbf{C},\mathbf{A}}_{\odot \mathbf{c}_j}} \circ l = k_i \tag{A.184}$$

because the factorization of h_i through π_i is unique, and

$$l = \sum_i i_i^{C^{\mathbf{C},\mathbf{A}}_{\odot \mathbf{c}_j}} \circ p_i^{C^{\mathbf{C},\mathbf{A}}_{\odot \mathbf{c}_j}} \circ l = \sum_i i_i^{C^{\mathbf{C},\mathbf{A}}_{\odot \mathbf{c}_j}} \circ k_i = h^*.$$

✠

Lemma A.6.4. *Let* $\mathbf{A}$ *be an ab-category with finite biproducts,* $F_i \in \mathcal{O}(\mathbf{A}^{\mathbf{C}})$ *for* $i = 1, \ldots, n$ *and suppose that* $(\mathbf{c}_1, \pi_i)$ *is a direct limit for* F_i *for* $i = 1, \ldots, n$*. Then* $(\odot \mathbf{c}_i, \odot \pi_i)$ *is a direct limit for* $\odot F_i$*.*

Proof. Analogous as proof to Lemma A.6.3. ✠

Lemma A.6.5. *Do we really need this? If* F *and* G *are functors such that* $\varprojlim F$ *and* $\varprojlim G$ *exist,* $(\mathbf{a}, f)$ *and* $(\mathbf{b}, g)$ *inverse targets for* F *and* G *that factor through* f^* *and* g^**, then* $f \odot g$ *factors through* $f^* \odot g^*$*.*

Proof. We have

$$f \odot g = (p^{\varprojlim F} \circ f^*) \odot (p^{\varprojlim G} \circ f^*) = (p^{\varprojlim F} \odot p^{\varprojlim G}) \circ (f^* \odot g^*).$$

✠

Lemma A.6.6. *Do we really need this? If* F *and* G *are functors such that* $\varinjlim F$ *and* $\varinjlim G$ *exist,* $(\mathbf{a}, f)$ *and* $(\mathbf{b}, g)$ *direct targets for* F *and* G *that factor through* f^* *and* g^**, then* $f \odot g$ *factors through* $f^* \odot g^*$*.*

Proof. We have

$$f \odot g = (f^* \circ i^{\varinjlim F}) \odot (g^* \circ i^{\varinjlim G}) = (f^* \odot g^*) \circ (i^{\varinjlim F} \odot i^{\varinjlim G}).$$

✠

Lemma A.6.7. *Let* $\mathbf{C}$ *be a category,* $\mathbf{A}$ *an ab-category with finite biproducts, for* $i = 1, \ldots, n$ *let* $F_i, G_i \in \mathcal{O}(\mathbf{A}^{\mathbf{C}})$*,* $(\mathbf{c}_i, \pi_i)$ *and* $(\mathbf{d}_i, \tau_i)$ *inverse limits for* F_i *and* G_i*,* $f_i : F_i \to G_i$*,* $h_i : C^{\mathbf{C},\mathbf{A}}_{\mathbf{c}_i} \to C^{\mathbf{C},\mathbf{A}}_{\mathbf{d}_i}$ *an inverse limit of* f_i*. Then* $\odot h_i$ *is an inverse limit of* $\odot f_i$*.*

Proof. We have

$$\bigodot \mathrm{f}_i \circ \bigodot \pi_i = \bigodot (\mathrm{f}_i \circ \pi_i) = \bigodot (\tau_i \circ \mathrm{h}_i) = \bigodot \tau_i \circ \bigodot \mathrm{h}_i. \tag{A.185}$$

✠

Lemma A.6.8. *Let* $\mathbf{C}$ *be a category,* $\mathbf{A}$ *an ab-category with finite biproducts, for* $i = 1, \dots, n$ *let* $\mathrm{F}_i, \mathrm{G}_i \in \mathcal{O}(\mathbf{A}^{\mathbf{C}})$*,* $(\mathbf{c}_i, \pi_i)$ *and* $(\mathbf{d}_i, \tau_i)$ *direct limits for* F_i *and* G_i*,* $\mathrm{f}_i : \mathrm{F}_i \to \mathrm{G}_i$*,* $\mathrm{h}_i : \mathrm{C}^{\mathbf{C},\mathbf{A}}_{\mathbf{c}_i} \to \mathrm{C}^{\mathbf{C},\mathbf{A}}_{\mathbf{d}_i}$ *a direct limit of* f_i*. Then* $\bigodot \mathrm{h}_i$ *is a direct limit of* $\bigodot \mathrm{f}_i$*.*

Proof. Analogous as proof of Lemma A.6.7. ✠

Lemma A.6.9. *Let* $\mathbf{A}$ *be an ab-category. If for* $i = 1, \dots, n$ f_i *is an inverse equalizer for* g_i*,* h_i*, and* $\bigodot f_i$*,* $\bigodot g_i$*,* $\bigodot h_i$ *exist, then* $\bigodot f_i$ *is an inverse equalizer for* $\bigodot g_i$*,* $\bigodot h_i$*.*

Proof. We have

$$\bigodot g_i \circ \bigodot f_i = \bigodot (g_i \circ f_i) = \bigodot (h_i \circ f_i) = \bigodot h_i \circ \bigodot f_i. \tag{A.186}$$

For $i = 1, \dots, n$ let $f_i : \mathbf{a}_i \to \mathbf{b}_i$, $g_i : \mathbf{b}_i \to \mathbf{c}_i$, $h_i : \mathbf{b}_i \to \mathbf{c}_i$. If $f : \mathbf{a} \to \bigodot \mathbf{b}_i$ is such that $(\bigodot g_i) \circ f = (\bigodot h_i) \circ f$, then

$$\sum_i i_i^{\bigodot \mathbf{c}_j} g_i p_i^{\bigodot \mathbf{b}_j} f = \sum_i i_i^{\bigodot \mathbf{c}_j} h_i p_i^{\bigodot \mathbf{b}_j} f \tag{A.187}$$

whence, composing on the left by $p_h^{\bigodot \mathbf{c}_j}$

$$g_h p_h^{\bigodot \mathbf{b}_j} f = h_h p_h^{\bigodot \mathbf{b}_j} f. \tag{A.188}$$

For $h = 1, \dots, n$ there are unique morphisms f_h^* such that $p_h^{\bigodot \mathbf{b}_i} f = f_h f_h^*$. Set $f^* = \sum i_i^{\bigodot \mathbf{a}_j} f_i^*$; then

$$\bigodot f_i \circ f^* = \sum i_i^{\bigodot \mathbf{b}_j} f_i p_i^{\bigodot \mathbf{a}_j} \circ \sum i_i^{\bigodot \mathbf{a}_j} f_i^* = \sum i_i^{\bigodot \mathbf{b}_j} f_i f_i^* = \sum i_i^{\bigodot \mathbf{b}_j} p_i^{\bigodot \mathbf{b}_j} f = f. \tag{A.189}$$

If $\bigodot f_i \circ f^{**} = f$, then

$$\sum_i i_i^{\bigodot \mathbf{b}_j} f_i p_i^{\bigodot \mathbf{a}_j} f^{**} = f \tag{A.190}$$

thus

$$p_h^{\bigodot \mathbf{a}_j} f = f_h p_h^{\bigodot \mathbf{a}_j} f^{**} = f_h f_h^* \tag{A.191}$$

and, since the factorisations of the $p_h^{\bigodot \mathbf{a}_j} f$ are unique

$$p_h^{\bigodot \mathbf{a}_j} f^{**} = f_h^* \tag{A.192}$$

whence

$$f^{**} = \sum_h i_h^{\bigodot \mathbf{a}_j} p_h^{\bigodot \mathbf{a}_j} f^{**} = \sum i_i^{\bigodot \mathbf{a}_j} f_h^* = f^*. \tag{A.193}$$

✠

A.7 Abelian categories

Definition A.7.1. A ***preadditive category*** is a category in which every hom-set is an abelian group.

Definition A.7.2. An ***additive category*** is a preadditive category which has a null object.

Definition A.7.3. An ***abelian category*** is an additive category $\mathbf{A}$ in which the following conditions are satisfied

1. $\mathbf{A}$ has binary biproducts.
2. Every morphism of $\mathbf{A}$ has a kernel and a cokernel.
3. Every monomorphism of $\mathbf{A}$ is a kernel and every epimorphism of $\mathbf{A}$ is a cokernel.

Proposition A.7.1. *Let* $\mathbf{C}$ *be a category. If* $f \in \overline{\mathbf{C}}(\mathbf{a}, \mathbf{b})$, $\ker(f)$ *and* $\operatorname{cok}(\ker(f))$ *exist, then* $\ker(f)$ *is a kernel of* $\operatorname{cok}(\ker(f))$. *That is, if a morphism is a kernel and has a cokernel, then it is a kernel of any of its cokernels.*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$, $h : \mathbf{d} \to \mathbf{a}$ a kernel of f, $k : \mathbf{a} \to \mathbf{e}$ a cokernel of h. Since $f \circ h = \mathbf{0}^{\mathbf{d}}_{\mathbf{b}}$, f factors uniquely as $f = f^* \circ k$; if $g : \mathbf{c} \to \mathbf{a}$ is such that $k \circ g = \mathbf{0}^{\mathbf{c}}_{\mathbf{e}}$, then $f \circ g = \mathbf{0}^{\mathbf{c}}_{\mathbf{b}}$ also holds, so g factor uniquely as $g = h \circ g^*$. ✠

Proposition A.7.2. *Let* $\mathbf{C}$ *be a category. If* $f \in \overline{\mathbf{C}}(\mathbf{a}, \mathbf{b})$, $\operatorname{cok}(f)$ *and* $\ker(\operatorname{cok}(f))$ *exist, then* $\operatorname{cok}(f)$ *is a cokernel of* $\ker(\operatorname{cok}(f))$. *That is, if a morphism is a cokernel and has a kernel, then it is a cokernel of any of its kernels.*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$, $h : \mathbf{b} \to \mathbf{d}$ be a cokernel of f, $k : \mathbf{e} \to \mathbf{b}$ a kernel of h. Since $h \circ f = \mathbf{0}^{\mathbf{a}}_{\mathbf{d}}$, f factors uniquely as $f = k \circ f^*$; if $g : \mathbf{b} \to \mathbf{c}$ is such that $g \circ k = \mathbf{0}^{\mathbf{e}}_{\mathbf{c}}$, then $g \circ f = \mathbf{0}^{\mathbf{a}}_{\mathbf{c}}$ also holds, so g factor uniquely as $g = g^* \circ h$. ✠

Proposition A.7.3. *Let* $\mathbf{A}$ *be an abelian category,* $f \in \mathcal{M}(\mathbf{A})$. *If* f *is a monomorphism then* $f = \ker(\operatorname{cok}(f))$, *if* f *is an epimorphism then* $f = \operatorname{cok}(\ker(f))$.

Proof. Straightforward from Definition A.7.3 and Propositions A.7.1, A.7.2. ✠

Lemma A.7.1. *Let* $\mathbf{C}$ *be a category with zero morphisms,* $f \in \mathcal{M}(\mathbf{C})$. *If* f *is a kernel of a zero morphism then it is an isomorphism, if* f *is a cokernel of a zero morphism then it is an isomorphism.*

Proof. Suppose $f : \mathbf{k} \to \mathbf{a}$ is a kernel of $\mathbf{0}^{\mathbf{a}}_{\mathbf{b}}$. Since $\mathbf{0}^{\mathbf{a}}_{\mathbf{b}} \operatorname{id}_{\mathbf{a}} = \mathbf{0}^{\mathbf{a}}_{\mathbf{b}}$, then $\operatorname{id}_{\mathbf{a}}$ factors uniquely as $\operatorname{id}_{\mathbf{a}} = fg$. Thus $f = fgf$, and $gf = \operatorname{id}_{\mathbf{k}}$ because f is a monomorphism. The proof of the second part is analogous. ✠

Proposition A.7.4. *In a category with zero morphisms a morphism that is both a monomorphism and an epimorphism is an isomorphism.*

Proof. If $f : \mathbf{a} \to \mathbf{b}$ is a monomorphism, it is a kernel of a morphism $g : \mathbf{b} \to \mathbf{c}$, so $g \circ f = \mathbf{0}^{\mathrm{a}}_{\mathrm{c}}$, which implies $g = \mathbf{0}^{\mathrm{b}}_{\mathrm{c}}$ if f is also an epimorphism. By Lemma A.7.1 f is an isomorphism. ✠

Lemma A.7.2. *An abelian category has binary inverse equalisers.*

Proof. If $f_1, f_2 : \mathbf{a} \to \mathbf{b}$ it is straightforward to prove that a kernel of $f_1 - f_2$ is an inverse equaliser of f_1, f_2. ✠

Lemma A.7.3. *An abelian category has pullbacks.*

Proof. Straightforward from Proposition A.3.1 and Lemma A.7.2 and an abelian category having binary biproducts. ✠

Proposition A.7.5. *If* $\mathbf{A}$ *is an abelian category, so is* $\mathbf{A}^{\mathbf{J}}$ *for any category* $\mathbf{J}$.

Proof. Let $\mathrm{F}, \mathrm{G} \in \mathcal{O}(\mathbf{A}^{\mathbf{J}})$ and $\alpha, \beta \in \overline{\mathbf{A}^{\mathbf{J}}}(\mathrm{F}, \mathrm{G})$. Define the natural transformation $\alpha + \beta$ for each $\mathbf{j} \in \mathcal{O}(\mathbf{J})$ by

$$(\alpha + \beta)_{\mathrm{j}} = \alpha_{\mathrm{j}} + \beta_{\mathrm{j}}.$$

It is clear that thus every hom-set $\overline{\mathbf{A}^{\mathbf{J}}}(\mathrm{F}, \mathrm{G})$ is an additive group.
If $\mathbf{0}$ is a null object of $\mathbf{A}$, the functor $\mathrm{N} : \mathbf{J} \to \mathbf{A}$ defined by

$$\mathrm{N}(\mathbf{j}) = \mathbf{0} \quad \mathbf{j} \in \mathcal{O}(\mathbf{J})$$
$$\mathrm{N}(f) = \mathbf{0}^{\mathrm{o}}_{\mathrm{o}} \quad f \in \mathcal{M}(\mathbf{J})$$

is a null object in $\mathbf{A}^{\mathbf{J}}$.
$\mathbf{A}^{\mathbf{J}}$ has binary biproducts by A.6.7.
By Proposition A.2.35 every morphism in $\mathbf{A}^{\mathbf{J}}$ has a kernel, and by Proposition A.2.24 every morphism in $\mathbf{A}^{\mathbf{J}}$ has a cokernel.
If $\mathrm{f} \in \mathcal{M}(\mathbf{A}^{\mathbf{J}})$ is a monomorphism, by Theorem A.2.2 every component f_{j} is a monomorphism, therefore it is a kernel, and by Proposition A.7.1 it is a kernel of the component c_{j} of a cokernel c of f. Then f is a kernel of c. If $\mathrm{f} \in \mathcal{M}(\mathbf{A}^{\mathbf{J}})$ is an epimorphism, by Theorem *(to do, analogous to Theorem A.2.2 for direct limits)* every component f_{j} is an epimorphism, therefore it is a cokernel, and by Proposition A.7.2 it is a cokernel of the component k_{j} of a kernel k of f. Then f is a cokernel of k. ✠

Definition A.7.4. Let $\mathbf{C}$ be a category, $f \in \mathcal{M}(\mathbf{C})$. An ***image*** of f is a monomorphism m such that

- there is $e \in \mathcal{M}(\mathbf{C})$ such that $f = m \circ e$
- if $f = m' \circ e'$ and m' is a monomorphism then there is $v \in \mathcal{M}(\mathbf{C})$ such that $m = m' \circ v$.

Proposition A.7.6. *If* $\mathbf{C}$ *is a category,* $f \in \mathcal{M}(\mathbf{C})$ *and* m *is an image of* f*, then there is a unique* $e \in \mathcal{M}(\mathbf{C})$ *such that* $f = m \circ e$*.*

Proof. Straightforward from the fact that m is a monomorphism. ✠

Proposition A.7.7. *Two morphisms are images of the same morphism* f *if and only if they belong to the same subobject of* $\mathrm{cod}(f)$*.*

Proof. Straightforward from definition. ✠

Notation A.7.1. For a morphism f we will denote by $\mathfrak{img}(f)$ the equivalence class of all the images of f, which is a subobject of $\mathrm{cod}(f)$. Thus $m \in \mathfrak{img}(f)$ will mean that m is an image of f. We will also write $\mathrm{img}(f)$ for any element of $\mathfrak{img}(f)$, when there will be no need to specify further.

Lemma A.7.4. *If* f *is a monomorphism then* $f \in \mathfrak{img}(f)$*.*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$. We have $f = f \circ \mathrm{id}_{\mathbf{a}}$, and if m is a monomorphism such that $f = m \circ e$, then this is the unique factorisation of f through m. Since f is a monomorphism $f \in \mathfrak{img}(f)$. ✠

Proposition A.7.8. *If* $\mathbf{A}$ *is an abelian category, for any* $f \in \mathcal{M}(\mathbf{A})$ $\mathfrak{img}(f) = \mathfrak{ker}(\mathfrak{cok}(f))$*.*

Proof. Being $\mathfrak{img}(f)$ and $\mathfrak{ker}(\mathfrak{cok}(f))$ equivalence classes, it is enough to prove that there is a $\mathrm{cok}(\mathrm{ker}(f))$ in $\mathfrak{img}(f)$.
Let $f : \mathbf{a} \to \mathbf{b}$. Then f factors as $f = \mathrm{ker}(\mathrm{cok}(f)) \circ e$ because $\mathrm{cok}(f) \circ f = \mathbf{0}^{\mathbf{a}}_{\mathrm{Cok}(f)}$, and $\mathrm{ker}(\mathrm{cok}(f))$ is a monomorphism. We need to prove that if f factors as $f = g \circ h$ and g is a monomorphism, then $\mathrm{ker}(\mathrm{cok}(f))$ factors through g. By Proposition A.7.3 $g = \mathrm{ker}(\mathrm{cok}(g))$, so we will prove this by proving that $\mathrm{cok}(g) \circ \mathrm{ker}(\mathrm{cok}(f)) = \mathbf{0}^{\mathrm{Ker}(\mathrm{cok}(f))}_{\mathrm{Cok}(g)}$.
We have $\mathrm{cok}(g) \circ f = \mathrm{cok}(g) \circ g \circ h = \mathbf{0}^{\mathbf{a}}_{\mathrm{Cok}_g}$, thus $\mathrm{cok}(g)$ factors as $\mathrm{cok}(g) = m \circ \mathrm{cok}(f)$. It follows that $\mathrm{cok}(g) \circ \mathrm{ker}(\mathrm{cok}(f)) = m \circ \mathrm{cok}(f) \circ \mathrm{ker}(\mathrm{cok}(f)) = \mathbf{0}^{\mathrm{Ker}(\mathrm{cok}(f))}_{\mathrm{Cok}(g)}$ so indeed $\mathrm{ker}(\mathrm{cok}(f))$ factors through $\mathrm{ker}(\mathrm{cok}(g))$, that is, through g. This proves that $\mathrm{ker}(\mathrm{cok}(f)) \in \mathfrak{img}(f)$. ✠

Definition A.7.5. Let $\mathbf{C}$ be a category, $f \in \mathcal{M}(\mathbf{C})$. A ***coimage*** of f is an epimorphism e such that

- there is $m \in \mathcal{M}(\mathbf{C})$ such that $f = m \circ e$
- if $f = m' \circ e'$ and e' is an epimorphism then there is $v \in \mathcal{M}(\mathbf{C})$ such that $e = v \circ e'$.

Proposition A.7.9. *If* $\mathbf{C}$ *is a category,* $f \in \mathcal{M}(\mathbf{C})$ *and* e *is a coimage of* f*, then there is a unique* $m \in \mathcal{M}(\mathbf{C})$ *such that* $f = m \circ e$*.*

Proof. Straightforward from the fact that m is an epimorphism. ✠

Proposition A.7.10. *Two morphisms are coimages of the same morphism* f *if and only if they belong to the same superobject of* $\operatorname{dom}(f)$*.*

Proof. Straightforward from definition. ✠

Notation A.7.2. For a morphism f we will denote by $\mathfrak{coi}(f)$ the equivalence class of all the coimages of f, and by $\operatorname{coi}(f)$ any of its members, when that does not give rise to any confusion.

Lemma A.7.5. *If* f *is an epimorphism then* $f \in \mathfrak{coi}(f)$*.*

Proof. Let $f : \mathbf{a} \to \mathbf{b}$. We have $f = \mathrm{id}_{\mathbf{b}} \circ f$, and if e is an epimorphism such that $f = m \circ e$, then this is the unique factorisation of f through e. Since f is an epimorphism $f \in \mathfrak{coi}(f)$. ✠

Proposition A.7.11. *If* $\mathbf{A}$ *is an abelian category, for any* $f \in \mathcal{M}(\mathbf{A})$ $\mathfrak{coi}(f) = \mathfrak{cok}(\mathfrak{ker}(f))$*.*

Proof. Being $\mathfrak{coi}(f)$ and $\mathfrak{cok}(\mathfrak{ker}(f))$ equivalence classes, it is enough to prove that there is a $\operatorname{cok}(\ker(f))$ in $\mathfrak{coi}(f)$.
Let $f : \mathbf{a} \to \mathbf{b}$. Then f factors as $f = e \circ \operatorname{cok}(\ker(f))$ because $f \circ \ker(f) = \mathbf{0}_{\mathbf{b}}^{\operatorname{Ker}(f)}$, and $\operatorname{cok}(\ker(f))$ is a monomorphism. We need to prove that if f factors as $f = g \circ h$ and h is an epimorphism, then $\operatorname{cok}(\ker(f))$ factors through h. By Proposition A.7.3 $g = \operatorname{cok}(\ker(g))$, so we will prove this by proving that $\operatorname{cok}(\ker(f)) \circ \ker(h) = \mathbf{0}_{\operatorname{Cok}(\ker(f))}^{\operatorname{Ker}(h)}$.
We have $f \circ \ker(h) = e \circ h \circ \ker(h) = \mathbf{0}_{\mathbf{b}}^{\operatorname{Ker}(h)}$, thus $\ker(h)$ factors as $\ker(h) = ker(f) \circ m$. It follows that $\operatorname{cok}(\ker(f)) \circ \ker(h) = \operatorname{cok}(\ker(f)) \circ \ker(h) \circ \ker(g) \circ m = \mathbf{0}_{\operatorname{Cok}(ker(f))}^{\operatorname{Ker}(h)}$ so indeed $\operatorname{cok}(\ker(f))$ factors through $\operatorname{cok}(\ker(h))$, that is, through h. This proves that $\operatorname{cok}(\ker(f)) \in \mathfrak{coi}(f)$. ✠

Lemma A.7.6. *Let* $\mathbf{A}$ *be an abelian category,* $f \in \overline{\mathbf{A}}(\mathbf{a}, \mathbf{b})$*,* $g \in \overline{\mathbf{A}}(\mathbf{b}, \mathbf{c})$*. Then* $\mathfrak{img}(f) = \mathfrak{ker}(g)$ *if and only if* $\mathfrak{cok}(f) = \mathfrak{coi}(g)$*.*

Proof. We have

$$\begin{aligned} \mathfrak{img}(f) = \mathfrak{ker}(g) &\iff \mathfrak{ker}(\mathfrak{cok}(g)) = \mathfrak{ker}(g) \\ &\iff \mathfrak{cok}(\mathfrak{ker}(\mathfrak{cok}(f))) = \mathfrak{cok}(\mathfrak{ker}(g)) \\ &\iff \mathfrak{cok}(f) = \mathfrak{coi}(g). \end{aligned}$$

✠

Definition A.7.6. A diagram of morphisms

$$\cdots\ \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{b}\ \cdots$$

is ***exact*** at $\mathbf{b}$ if and only if $\mathfrak{img}(f) = \mathfrak{ker}(g)$ or equivalently $\mathfrak{cok}(f) = \mathfrak{coi}(g)$.

Lemma A.7.7. *In a category with zero morphisms for any two objects* $\mathbf{a}$ *and* $\mathbf{b}$ $\mathfrak{cok}(\mathbf{0}^{\mathbf{a}}_{\mathbf{b}}) = [\mathrm{id}_{\mathbf{b}}]_{\mathbf{b}}$.

Proof. Straightforward from the definition of cokernel. ✠

Lemma A.7.8. *In a category with zero morphisms for any two objects* $\mathbf{a}$ *and* $\mathbf{b}$ $\mathfrak{img}(\mathbf{0}^{\mathbf{a}}_{\mathbf{b}}) = [\mathbf{0}^{\mathbf{0}}_{\mathbf{b}}]_{\mathbf{b}}$.

Proof. Straightforward from the definition of image. ✠

Proposition A.7.12. *In an abelian category the diagram of morphisms*

$$\mathbf{0} \longrightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \tag{A.194}$$

is exact at $\mathbf{a}$ *if and only if* f *is a monomorphism.*

Proof. The diagram (A.194) is exact at $\mathbf{a}$, by Lemma A.7.8, if and only if $\mathfrak{ker}(f) = [\mathbf{0}^{\mathbf{0}}_{\mathbf{a}}]$, so if and only if f is a monomorphism. ✠

Lemma A.7.9. *In a category with zero morphisms for any two objects* $\mathbf{a}$ *and* $\mathbf{b}$ $\mathfrak{ker}(\mathbf{0}^{\mathbf{a}}_{\mathbf{b}}) = [\mathrm{id}_{\mathbf{a}}]^{\mathbf{a}}$.

Proof. Straightforward from the definition of kernel. ✠

Lemma A.7.10. *In a category with zero morphisms for any two objects* $\mathbf{a}$ *and* $\mathbf{b}$ $\mathfrak{coi}(\mathbf{0}^{\mathbf{a}}_{\mathbf{b}}) = [\mathbf{0}^{\mathbf{a}}_{\mathbf{0}}]^{\mathbf{a}}$.

Proof. Straightforward from the definition of image. ✠

Proposition A.7.13. *In an abelian category the diagram of morphisms*

$$\mathbf{a} \xrightarrow{f} \mathbf{b} \longrightarrow \mathbf{0} \tag{A.195}$$

is exact at $\mathbf{b}$ *if and only if* f *is an epimorphism.*

Proof. The diagram (A.195) is exact at $\mathbf{b}$, by Lemma A.7.10, if and only if $\mathfrak{cok}(f) = [\mathbf{0}^{\mathbf{b}}_{\mathbf{0}}]$, so if and only if f is am epimorphism. ✠

Definition A.7.7. In a category with a null object $\mathbf{0}$ a diagram of morphisms

$$\mathbf{0} \longrightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \longrightarrow \mathbf{0}$$

is a ***short exact sequence of morphisms*** if it is exact at $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Proposition A.7.14. *In an abelian category the diagram of morphisms*

$$\mathbf{0} \longrightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \longrightarrow \mathbf{0} \tag{A.196}$$

is a short exact sequence if and only if $f \in \mathfrak{ker}(g)$ *and* $g \in \mathfrak{cok}(f)$*.*

Proof. By Proposition A.7.12 the diagram (A.196) is exact at **a** if and only if f is monic. By Proposition A.7.13 the diagram (A.196) is exact at **c** if and only if g is epic. Thus $f \in \mathfrak{img}(f)$ and $g \in \mathfrak{coi}(g)$, and the diagram (A.196) is exact at **b** if and only if $f \in \mathfrak{ker}(g)$ or $g \in \mathfrak{cok}(f)$. ✠

Definition A.7.8. In a category with zero morphisms a diagram of morphisms

$$\mathbf{0} \longrightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c}$$

is a ***short left exact sequence of morphisms*** if it is exact at **a** and **b**.

Proposition A.7.15. *In an abelian category the diagram of morphisms*

$$\mathbf{0} \longrightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \tag{A.197}$$

is a short left exact sequence if and only if $f \in \mathfrak{ker}(g)$*.*

Proof. By Proposition A.7.12 the diagram (A.197) is exact at **a** if and only if f is monic, that is if and olny if $f \in \mathfrak{img}(f)$, and so the diagram (A.197) is exact at **b** if and only if $f \in \mathfrak{ker}(g)$. ✠

Definition A.7.9. In a category with a null object **0** a diagram of morphisms

$$\mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \longrightarrow \mathbf{0}$$

is a ***short right exact sequence of morphisms*** if it is exact at **b** and **c**.

Proposition A.7.16. *In an abelian category the diagram of morphisms*

$$\mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \longrightarrow \mathbf{0} \tag{A.198}$$

is a short right exact sequence if and only if $g \in \mathfrak{cok}(f)$*.*

Proof. By Proposition A.7.13 the diagram (A.198) is exact at **c** if and only if g is epic, that is if and olny if $g \in \mathfrak{coi}(g)$, and the diagram (A.198) is exact at **b** if and only if $g \in \mathfrak{cok}(g)$. ✠

Definition A.7.10. A functor $\mathrm{T} : \mathbf{A} \to \mathbf{B}$ between abelian categories is

- ***left exact*** if it preserves finite inverse limits, that is, if for any finite category $\mathbf{I}$, any functor $\mathrm{F} : \mathbf{I} \to \mathbf{A}$ and any inverse limit $(\mathbf{l}, \mu)$ of F, $(\mathrm{T}(\mathbf{l}), \mathrm{T}(\mu))$ is an inverse limit of TF.

- ***right exact*** if it preserves finite direct limits, that is, if for any finite category $\mathbf{I}$, any functor $\mathrm{F} : \mathbf{I} \to \mathbf{A}$ and any direct limit $(\mathbf{l}, \mu)$ of F, $(\mathrm{T}(\mathbf{l}), \mathrm{T}(\mu))$ is a direct limit of TF.

- ***exact*** if it is both left and right exact.

Remark A.7.1. A left exact functor $\mathrm{T} : \mathbf{A} \to \mathbf{B}$ between abelian categories preserves in particular kernels, which can be stated in the form $\mathrm{T}(\ker(f)) = \ker(\mathrm{T}(f))$, and thus it preserves short left exact sequences, that is, if

$$\mathbf{0} \longrightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c}$$

is exact, then

$$\mathbf{0} \longrightarrow \mathrm{T}(\mathbf{a}) \xrightarrow{\mathrm{T}(f)} \mathrm{T}(\mathbf{b}) \xrightarrow{\mathrm{T}(g)} \mathrm{T}(\mathbf{c})$$

is also exact.
A right exact functor $\mathrm{T} : \mathbf{A} \to \mathbf{B}$ between abelian categories preserves in particular cokernels, which can be stated in the form $\mathrm{T}(\mathrm{cok}(f)) = \mathrm{cok}(\mathrm{T}(f))$, and thus it preserves short right exact sequences, that is, if

$$\mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \longrightarrow \mathbf{0}$$

is exact, then

$$\mathrm{T}(\mathbf{a}) \xrightarrow{\mathrm{T}(f)} \mathrm{T}(\mathbf{b}) \xrightarrow{\mathrm{T}(g)} \mathrm{T}(\mathbf{c}) \longrightarrow \mathbf{0}$$

is also exact.

Proposition A.7.17. *A functor between abelian categories is left exact if and only if it is additive and preserves kernels.*

Proof. If a functor is left exact it preserve in particular finite inverse products, and so it preserves binary biproducts, and by Proposition A.6.5 it is additive.
If a functor is additive By Proposition A.6.5 it preserves binary biproducts, and if it preserves kernels it preserves also binary inverse equalisers. Then by Theorem A.2.1 it preserves finite inverse limits. ✠

Proposition A.7.18. *A functor between abelian categories is right exact if and only if it is additive and preserves cokernels.*

Proof. If a functor is right exact it preserve in particular finite direct products, and so it preserves binary biproducts, and by Proposition A.6.5 it is additive.
If a functor is additive By Proposition A.6.5 it preserves binary biproducts, and if it preserves cokernels it preserves also binary direct equalisers. Then by Theorem (TO DO!!!) it preserves finite direct limits. ✠

Lemma A.7.11. *If f is a kernel of $k \circ j$ and j is an isomorphism, then $j \circ f$ is a kernel of k.*

Proof. Surely $k \circ (j \circ f) = \mathbf{0}_{\mathrm{cod}(k)}^{\mathrm{dom}(f)}$. If $k \circ h = \mathbf{0}_{\mathrm{cod}(k)}^{\mathrm{dom}(h)}$, set $l = j^{-1} \circ h$; then $(k \circ j) \circ l = \mathbf{0}_{\mathrm{cod}(k)}^{\mathrm{dom}(l)}$, thus $l = f \circ h^*$, and $h = j \circ f \circ h^*$. If also $h = j \circ f \circ h^\circ$, then $h^\circ = h^*$ because $j \circ f$ is a monomorphism. ✠

Lemma A.7.12. *If f is a kernel of $i \circ k$ and i is an isomorphism, then f is a kernel of k.*

Proof. Since $i \circ k \circ f = \mathbf{0}_{\mathrm{cod}(i)}^{\mathrm{dom}(f)}$ then $k \circ f = \mathbf{0}_{\mathrm{cod}(k)}^{\mathrm{dom}(f)}$. If $k \circ g = \mathbf{0}_{\mathrm{cod}(k)}^{\mathrm{dom}(g)}$, then also $i \circ k \circ g = \mathbf{0}_{\mathrm{cod}(i)}^{\mathrm{dom}(g)}$, thus $g = f \circ g^*$. If also $g = f \circ g^\circ$ then $g^\circ = g^*$ because f is a monomorphism. ✠

Lemma A.7.13. *If f is a cokernel of $j \circ k$ and j is an isomorphism, then $f \circ j$ is a cokernel of k.*

Proof. Surely $(f \circ j) \circ k = \mathbf{0}_{\mathrm{cod}(f)}^{\mathrm{dom}(k)}$. If $h \circ k = \mathbf{0}_{\mathrm{cod}(h)}^{\mathrm{dom}(k)}$, set $l = h \circ j^{-1}$; then $l \circ (j \circ k) = \mathbf{0}_{\mathrm{cod}(l)}^{\mathrm{dom}(k)}$, thus $l = h^* \circ f$, and $h = h^* \circ f \circ j$. If also $h = h^\circ \circ f \circ j$, then $h^\circ = h^*$ because $f \circ j$ is an epimorphism. ✠

Lemma A.7.14. *If f is a cokernel of $k \circ i$ and i is an isomorphism, then f is a cokernel of k.*

Proof. Since $f \circ k \circ i = \mathbf{0}_{\mathrm{cod}(f)}^{\mathrm{dom}(i)}$ then $f \circ k = \mathbf{0}_{\mathrm{cod}(f)}^{\mathrm{dom}(k)}$. If $g \circ k = \mathbf{0}_{\mathrm{cod}(g)}^{\mathrm{dom}(k)}$, then also $g \circ k \circ i = \mathbf{0}_{\mathrm{cod}(g)}^{\mathrm{dom}(i)}$, thus $g = g^* \circ f$. If also $g = g^\circ \circ f$ then $g^\circ = g^*$ because f is an epimorphism. ✠

Lemma A.7.15. *If in the commutative diagram*

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \mathbf{a}_1 & \xrightarrow{f_1} & \mathbf{a}_2 & \xrightarrow{f_2} & \mathbf{a}_3 & \longrightarrow & \mathbf{0} \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} & & \\
\mathbf{0} & \longrightarrow & \mathbf{b}_1 & \xrightarrow{g_1} & \mathbf{b}_2 & \xrightarrow{g_2} & \mathbf{b}_3 & \longrightarrow & \mathbf{0}
\end{array}$$

the first row is exact and the vertical morphisms are isomorphisms, then the second row is also exact.

Proof. Since $f_1 = \ker(f_2) = \ker(i_3^{-1} \circ g_2 \circ i_2)$, from Lemma A.7.12 $f_1 = \ker(g_2 \circ i_2)$ and from Lemma A.7.11 $i_2 \circ f_1 = \ker(g_2)$, thus $g_1 \circ i_1 = \ker(g_2)$ and $g_1 = \ker(g_2)$.
Since $f_2 = \mathrm{cok}(f_1) = \mathrm{cok}(i_2^{-1} \circ g_1 \circ i_1)$, from Lemma A.7.14 $f_2 = \mathrm{cok}(i_2^{-1} \circ g_1)$ and from Lemma A.7.13 $f_2 \circ i_2^{-1} = \mathrm{cok}(g_1)$, thus $i_3^{-1} \circ g_2 = \mathrm{cok}(g_1)$ and $g_2 = \mathrm{cok}(g_1)$. ✠

Appendix B

Algebraic types and varieties of algebras

B.1 Algebraic types

Definition B.1.1. An ***algebraic type*** Ω is a pair (S, f) where S is a set and f is a map from S to $\mathcal{O}(\mathbf{Set})$. The set S is called the ***operation set of*** Ω and its elements are called the ***operation symbols of*** Ω, the map f is called the ***arity map of*** Ω and for $s \in \mathrm{S}$ the set $f(s)$ is called the ***arity of*** s.

Notation B.1.1. For an algebraic type Ω we will write $|\Omega|$ for its operation set and ar_Ω for its arity map.

Definition B.1.2. An algebraic type Ω is called ***finitary*** if for each $s \in |\Omega|$, $\mathrm{card}(\mathrm{ar}_\Omega(s)) < \omega$.
An algebraic type Ω is called ***conventional*** if $|\Omega|$ is a cardinal and for each $s \in |\Omega|$, $\mathrm{ar}_\Omega(s)$ is a cardinal.

Definition B.1.3. Let Ω be an algebraic type. An Ω***-algebra*** is a pair $(\mathrm{T}, (f_s)_{s\in|\Omega|})$ where T is a set and for each $s \in |\Omega|$ f_s is a map from $\mathrm{T}^{\mathrm{ar}_\Omega(s)}$ to T, that is, an operation of arity $\mathrm{ar}_\Omega(s)$ on T.

Notation B.1.2. For an Ω algebra $\mathbf{A} = (\mathrm{T}, (f_s)_{s\in|\Omega|})$ we will write $|\mathbf{A}|$ for T and for $s \in |\Omega|$ $s_\mathbf{A}$ for f_s.

Definition B.1.4. If **A**a and **B** are Ω-algebras, an ***homomorphism*** from **A** to **B** is a map $f : |\mathbf{A}| \to |\mathbf{B}|$ such that for each $s \in |\Omega|$ and each $(x_i)_{i\in\mathrm{ar}_\Omega(s)} \in |\mathbf{A}|^{\mathrm{ar}_\Omega(s)}$

$$f(s_\mathbf{A}((x_i)_{i\in\mathrm{ar}_\Omega(s)})) = s_\mathbf{B}\left((f(x_i))_{i\in\mathrm{ar}_\Omega(s)}\right). \tag{B.1}$$

Notation B.1.3. If Ω is an algebraic type, Ω**-Alg** is the category of Ω-algebras and homomorphisms between Ω-algebras.

Definition B.1.5. Let $\mathbf{A}$ be an Ω-algebra. A ***subalgebra of*** $\mathbf{A}$ is an Ω-algebra $\mathbf{B}$ such that $|\mathbf{B}| \subseteq |\mathbf{A}|$ and the inclusion map from $\mathbf{B}$ to $\mathbf{A}$ is a homomorphism.

Remark B.1.1. If $\mathbf{A}$ is an Ω-algebra the subalgebras of $\mathbf{A}$ correspond to the subsets of $|\mathbf{A}|$ which are closed under the operations of $\mathbf{A}$.

Notation B.1.4. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ we will write $\mathbf{B} \sqsubseteq \mathbf{A}$.

Remark B.1.2. The subalgebras of an Ω-algebra constitute a set partially ordered by the relation $\sqsubseteq$.

Notation B.1.5. The set of subalgebras of an Ω-algebra $\mathbf{A}$ will be noted by $\mathcal{S}(\mathbf{A})$.

Definition B.1.6. A ***homomorphic image of an*** Ω***-algebra*** $\mathbf{A}$ is an Ω-algebra $\mathbf{B}$ such that there exists a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ which is surjective as a map from $|\mathbf{A}|$ to $|\mathbf{B}|$.

Proposition B.1.1. *Let* $\mathbf{A}$ *be an* Ω*-algebra. For any* $\mathrm{S} \subseteq \mathcal{S}(\mathbf{A})$ *there is* $\mathbf{B} \sqsubseteq \mathbf{A}$ *such that* $|\mathbf{B}| = \bigcap_{\mathrm{C} \in \mathrm{S}} |\mathbf{C}|$.

Definition B.1.7. Let $\mathbf{A}$ be an Ω-algebra, $\mathrm{S} \subseteq \mathcal{S}(\mathbf{A})$. The subalgebra $\mathbf{B} \sqsubseteq \mathbf{A}$ such that $|\mathbf{B}| = \bigcap_{\mathrm{C} \in \mathrm{S}} |\mathbf{C}|$ is called the ***intersection subalgebra of*** S and noted by $\bigsqcap \mathrm{S}$.

Definition B.1.8. Let $\mathbf{A}$ be an Ω-algebra, $\mathrm{X} \subseteq |\mathbf{A}|$. The subalgebra of $\mathbf{A}$

$$\mathbf{A}^{\mathrm{X}} = \bigsqcap \{\mathrm{Y} \in \mathcal{S}(\mathbf{A}) | \mathrm{X} \subseteq \mathrm{Y}\} \tag{B.2}$$

is called the ***subalgebra of*** $\mathbf{A}$ ***generated by*** X.

Remark B.1.3. For an Ω-algebra $\mathbf{A}$ and $\mathrm{X} \subseteq |\mathbf{A}|$, $\mathbf{A}^{\mathrm{X}}$ is the smallest subalgebra of $\mathbf{A}$ whose set contains X.

Remark B.1.4. For an Ω-algebra $\mathbf{A}$, $\mathrm{X} \subseteq |\mathbf{A}|$ is the set of a subalgebra of $\mathbf{A}$ if and only if $\mathrm{X} = |\mathbf{A}^{\mathrm{X}}|$.

Definition B.1.9. We say that an Ω-algebra $\mathbf{A}$ is generated by $\mathrm{X} \subseteq |\mathbf{A}|$ if $\mathbf{A} = \mathbf{A}^{\mathrm{X}}$.

Remark B.1.5. $\mathbf{A} = \mathbf{A}^{\mathrm{X}}$ if and only if $\mathbf{A}$ is the only subalgebra of $\mathbf{A}$ whose set contains X.

Definition B.1.10. Let $f \in \overline{\mathbf{Set}}(\mathrm{I}, \mathcal{O}(\mathbf{Set}))$. The set

$$\bigtimes f = \left\{ x \in \overline{\mathbf{Set}}\big(\mathrm{I}, \cup \operatorname{img}(f)\big) \mid x(i) \in f(i) \right\} \tag{B.3}$$

is called the ***cartesian product of*** f.
For $i \in \mathrm{I}$ the maps

$$\begin{aligned} p_i^{\times f} :\ & \bigtimes f \to f(i) \\ & x \mapsto x(i) \end{aligned} \tag{B.4}$$

are called the ***projections of*** $\bigtimes f$.

Notation B.1.6. If $\mathbf{C}$ is a category by saying that $(\mathbf{c}_i)_{i\in \mathrm{I}}$ is a ***family of elements of*** $\mathbf{C}$ we understand that there is $f \in \overline{\mathbf{Set}}(\mathrm{I}, \mathcal{O}(\mathbf{C}))$ such that for $i \in \mathrm{I}$ $f(i) = \mathbf{c}_i$.
The cartesian product of a family of sets $(\mathrm{S}_i)_{i\in \mathrm{I}}$ will also be written as

$$\bigtimes_{i\in \mathrm{I}} \mathrm{S}_i \tag{B.5}$$

Lemma B.1.1. *If* $(\mathbf{A}_i)_{i\in \mathrm{I}}$ *is a small FAMILY of* Ω*-algebras, let* $(\mathrm{P}, (\pi)_{i\in \mathrm{I}}) \in \prod_{i\in \mathrm{I}}|\mathbf{A}_i|$. *There is an* Ω*-algebra* $\mathbf{P}$ *such that* $|\mathbf{P}| = \mathrm{P}$*, the* π_i *are* Ω*-algebra homomorphisms, and* $(\mathbf{P}, (\pi)_{i\in \mathrm{I}})$ *is an inverse product of the* $\mathbf{A}_i$ *in* Ω***-Alg****. For* $s \in |\Omega|$ *the operation* $s_{\mathbf{P}}$ *is defined by*

$$s_{\mathbf{P}}\big((x_i)_{i\in \mathrm{ar}(s)}\big) = s_{\mathbf{P}}\big(((x_{ij})_{j\in \mathrm{I}})_{i\in \mathrm{ar}(s)}\big) = \big(s_{\mathbf{A}}\big((x_{ij})_{i\in \mathrm{ar}(s)}\big)\big)_{j\in \mathrm{I}} . \tag{B.6}$$

Proof. Routine check. ✠

Lemma B.1.2. *Let* $h_1, h_2 \in \overline{\Omega\text{-}\boldsymbol{Alg}}(\mathbf{A}, \mathbf{B})$*, and let* (E, i) *be the standard equalizer of* h_1 *and* h_2 *as set maps, that is*

$$\mathrm{E} = \{x \in |\mathbf{A}| | h_1(x) = h_2(x)\} \tag{B.7}$$

and

$$\begin{aligned} i : \mathrm{E} &\to |\mathbf{A}| \\ x &\mapsto x. \end{aligned} \tag{B.8}$$

Then $\mathrm{E} = |\mathbf{A}^{\mathrm{E}}|$ *and* i *is an* Ω*-algebra homomorphism.*

Proof. Routine check. ✠

Appendix C

Properties of a universe

A set U is a universe if

1. $x \in y \in \mathrm{U} \Rightarrow x \in \mathrm{U}$
2. $x \in \mathrm{U} \wedge y \in \mathrm{U} \Rightarrow \{x, y\} \in \mathrm{U}$, $<x, y> \in \mathrm{U}$, $x \times y \in \mathrm{U}$
3. $x \in \mathrm{U} \Rightarrow \mathcal{P}(x) \in \mathrm{U}$, $\cup x \in \mathrm{U}$
4. $\omega \in \mathrm{U}$
5. if $f : x \rightarrow y$ is surjective, $x \in \mathrm{U}$ and $y \subset \mathrm{U}$ then $y \in \mathrm{U}$

Appendix D

Symbols

- $\mathcal{U}_x$
- $\mathcal{U}_x^{\mathrm{o}}$
- $\mathbf{C}^{\mathrm{q}}$ quotient category of category $\mathbf{C}$
- $(a_s)_{s\in S}$: collection of objects a_s indexed by the set S
- **Set**: category of sets
- **Grp**: category of groups
- **Ab**: category of abelian groups
- $\mathbf{Alg}_\tau$: category of τ-algebras.
- $\mathbf{Alg}_{\tau,A}$: category of τ-algebras over τ-algebra A.
- $\mathbf{0}_{\mathbf{b}}^{\mathbf{a}}$: null morphism from $\mathbf{a}$ to $\mathbf{b}$.
- **Psh**: Category of presheaves
- $\mathbf{Psh}_X$: Category of presheaves on topological space X
- $\mathbf{Psh}_{(X,\mathbf{C})}$: Category of presheaves of category $\mathbf{C}$ on topological space X
- $\mathbf{Psh}^*_{X,\tau}$: Category of presheaves of τ-algebras on topological space X
- $\mathbf{Psh}^*_{X,\tau,A}$: Category of presheaves of τ-algebras over A on topological space X
- **Sh**: Category of sheaves
- $\mathbf{Sh}_X$: Category of sheaves on topological space X
- $\mathbf{Sh}_{X,\mathbf{C}}$: Category of sheaves of category $\mathbf{C}$ con topological space X

- $\mathbf{Sh}^*_{X,\tau}$: Category of sheaves of τ-algebras on topological space X
- $\mathbf{Sh}^*_{X,\tau,A}$: Category of sheaves of τ-algebras over A on topological space X
- $\mathbf{SS}$: Category of sheaf spaces
- $\mathbf{SS}_X$: Category of sheaf spaces on topological space X
- $\mathbf{SS}_{X,\tau}$: Category of sheaf spaces of τ-algebras con topological space X
- $\mathbf{Sh}^*_{X,\tau,A}$: Category of sheaf spaces of τ-algebras over A on topological space X
- $\hat{X}$ category associated to topological space X
- $\hat{\phi}$ functor associated to continuous map ϕ
- X^S: underlying set f topological space X
- X^T: family of open sets of topological space X
- S^*: discrete category of set S
- $\mathbf{0}$: trivial subgroup $\{0\}$ of $\mathbb{Z}$, trivial abelian group
- $\mathrm{F_o}$: object function of functor F
- $\mathrm{F_m}$: morphism function of functor F
- $\Sigma(\mathcal{E}, U)$: set of sections of sheaf space $\mathcal{E}$ over open U
- $\overline{B}$: base τ-algebra of τ-algebra over A B
- α_B: structure map of τ-algebra over A B
- $[f]_{\mathbf{a}}$: subobject of **a** defined by morphism f
- $[f]^{\mathbf{a}}$: superobject of **a** defined by morphism f

Appendix E

Questions

- If two morphisms of presheaves agree on the stalk at x, do they agree on a neighbourhood of its?

Bibliography